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The prestressability problem of tensegrity structures: some analytical solutions

Cornel Sultan [a,*], Martin Corless [b], Robert E. Skelton [c]

[a] *Tensegra Inc., Norwood, MA 02062, USA*
[b] *School of Aeronautics and Astronautics, Purdue University, West Lafayette, IN 47907-1282, USA*
[c] *Department of Engineering Mechanics, University of California, La Jolla, CA 92093-0411, USA*

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Abstract

In this paper we formulate the general prestressability conditions for tensegrity structures. These conditions are expressed as a set of nonlinear equations and inequalities on the tendon tensions. Several examples of tensegrity structures for which the prestressability conditions can be analytically solved are then presented. © 2001 Published by Elsevier Science Ltd.

Keywords: Tensegrity structures; Prestressability; Analytical solutions

1. Introduction

The word *tensegrity* is an acronym, a contraction of *tensional integrity*. Fuller (1975) defines tensegrity as a "structural relationship principle in which structural shape is guaranteed by the interaction between a continuous network of members in tension and a set of members in compression".

Tensegrity structures are lattices that form finite networks depending on the arrangement of the vertices: tower-like structures, layered networks or crystalline type networks according to the number of spatial directions they develop. They consist of a set of *soft* members (for example elastic tendons), and a set of *hard* members (for example bars). A perspective view of a tensegrity structure composed of 24 tendons – the *soft* members – and six bars – the *hard* members – is given in Fig. 1.

Although the origins of tensegrity structures can be pin-pointed to 1927 (Snelson, 1996), the main investigations have been carried out during the last 40 years, with artistic work as the starting point. Tensegrity structures were looked upon from an engineering perspective for the first time by Fuller (1975). Geometrical investigations followed, most of them being reported in Fuller (1975) and Pugh (1976).

Approaches using mechanics have been developed recently and research in tensegrity structures turned into a more systematic and engineering oriented one, aimed at establishing the theoretical framework for

* Corresponding author. Fax: +1-617-427-1234.
E-mail address: csultan@tenseg.com (C. Sultan).

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Nomenclature

b	edge length of the base and top triangles
h	overlap
k_j	stiffness of the jth tendon
l	length of a bar
l_j	length of the jth tendon
l_{0_j}	rest length of the jth tendon
q	vector of generalized coordinates
x_{ij}, y_{ij}, z_{ij}	Cartesian coordinates of the mass center of the ij bar
$A(q)$	equilibrium matrix
C	force in a bar in a symmetrical prestressable configuration
D	length of a diagonal tendon in a symmetrical configuration
E	number of tendons
N	number of degrees of freedom
P	pretension coefficient
S	length of a saddle tendon in a symmetrical configuration
X, Y, Z	Cartesian coordinates of the mass center of the top
T	vector of tendon tensions
T_j	tension in the jth tendon
T_B	tension in a boundary tendon in a symmetrical prestressable configuration
T_D	tension in a diagonal tendon in a symmetrical prestressable configuration
T_S	tension in a saddle tendon in a symmetrical prestressable configuration
T_V	tension in a vertical tendon in a symmetrical prestressable configuration
T_T	tension in a top tendon in a symmetrical prestressable configuration
V	length of a vertical tendon in a symmetrical configuration
W	total virtual work
W_j	virtual work due to the jth tendon
δ	declination of a bar in a symmetrical configuration
α	azimuth of bar 11 in a symmetrical configuration
α_{ij}	azimuth of bar ij
δ_{ij}	declination of bar ij
δq	virtual displacement of the vector of generalized coordinates
δl_j	virtual change in length of the jth tendon
ψ, ϕ, θ	Euler angles of the top reference frame

the analysis and design of these structures. Among the researchers in tensegrity structures, Pellegrino (1990), Pellegrino and Calladine (1986), Motro (1992), Motro et al. (1986), and Hanaor (1988) have made important contributions toward further knowledge of the statics of these structures. Linear dynamic analysis results have been published by Motro (1986) and Furuya (1992). Nonlinear dynamics and control design studies have been reported by Skelton and Sultan (1997), Sultan (1999) and Sultan and Skelton (1997).

Industrial projects and proposals are beginning, ranging from tensegrity domes (Hanaor, 1992; Wang and Liu, 1996) to tensegrity sensors (Sultan and Skelton, 1998), space telescopes (Sultan et al., 1999), or flight simulators (Sultan et al., 2000). It is interesting to note that, recently, tensegrity structures have been

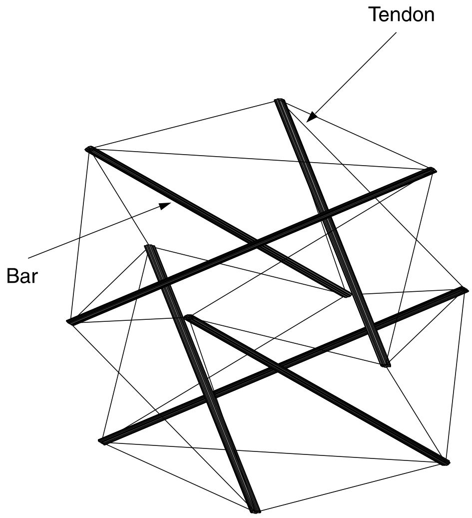

Fig. 1. A tensegrity structure.

proposed to explain how various types of cells (e.g. nerve cells, smooth muscles, etc.) resist shape distorsion (Ingber, 1993, 1998). Qualitative and quantitative results using a two stage tensegrity structure to model a cell's static properties and which are in agreement with biological experimental results, have been reported by Stamenovic et al. (1996) and Coughlin and Stamenovic (1997).

In this article we first derive the general prestressability conditions from the principle of virtual work. Next, certain classes of tensegrity structures are introduced and the prestressability conditions are investigated. For particular prestressable configurations these conditions are analytically solved.

2. General prestressability conditions

A *fundamental* question we have to answer when investigating tensegrity structures is as follows: under what conditions does a tensegrity structure yield an equilibrium configuration with all tendons in tension when no external forces and no external torques act? This property of tensegrity structures is called *prestressability*, the corresponding conditions are called *prestressability conditions*, and the corresponding equilibrium configurations are called *prestressable configurations*.

In the following we derive the general prestressability conditions using the principle of virtual work.

2.1. Mathematical modeling assumptions

Consider a tensegrity structure composed of E *elastic* tendons and R *rigid* bodies. We assume that all the joints of the system are affected atmost by *kinetic friction*. This means that the friction forces/torques acting at a joint are zero if the relative velocity between the elements in contact is zero and may be nonzero otherwise. Also, the tendons are affected atmost by *kinetic damping*. This means that the damping force introduced by a tendon is zero if the time derivative of its elongation is zero and may be nonzero otherwise. The system is assumed to be *holonomic*. All constraints are *scleronomic* and *bilateral*; in other words, they are not time dependent and they are not mathematically expressed as inequalities. The constraint forces are *workless*, which means that they do no work through a virtual displacement satisfying the geometric constraints. We neglect the forces exerted upon the structure by external force fields (for example, a gravitational field).

2.2. Derivation of the prestressability conditions

In order to derive the general prestressability conditions we apply the principle of virtual work: "*A necessary and sufficient condition for the static equilibrium of an initially motionless, scleronomic system which is subjected to workless bilateral constraints is that zero virtual work is done by the applied forces in moving through an arbitrary, reversible, virtual displacement satisfying the constraints*".

Assume that no external applied forces and torques act and that the structure is in equilibrium in a prestressable configuration. Let $\delta q = [\delta q_1 \;\; \delta q_2 \;\; \cdots \;\; \delta q_N]^{\mathrm{T}}$, represent a virtual displacement of the vector of *independent* generalized coordinates, $q = [q_1 \;\; q_2 \;\; \ldots \;\; q_N]^{\mathrm{T}}$, from its value at a prestressable configuration. Here N is the number of degrees of freedom of the system. Let us examine the virtual work of the applied forces acting on the structure.

Since the damping and friction forces and torques acting on the system are kinetic, these forces and torques are zero when the system is in equilibrium, hence they do no work in a virtual displacement. The only forces which do virtual work are the tensions in the tendons.

Due to the virtual displacement, the jth tendon experiences a virtual change in length of

$$\delta l_j = \sum_{i=1}^{N} \frac{\partial l_j}{\partial q_i} \delta q_i. \tag{1}$$

Here l_j is the length of the jth tendon. Throughout the virtual displacement, the tension in tendon j is constant, equal to its equilibrium value, T_j. The corresponding virtual work is

$$W_j = T_j \delta l_j. \tag{2}$$

Assuming E tendons, the total virtual work is given by

$$W = \sum_{j=1}^{E} W_j = \sum_{j=1}^{E} T_j \sum_{i=1}^{N} \frac{\partial l_j}{\partial q_i} \delta q_i = \sum_{i=1}^{N} \delta q_i \sum_{j=1}^{E} \frac{\partial l_j}{\partial q_i} T_j = \delta q^{\mathrm{T}} A(q) T, \tag{3}$$

where T is the vector of tendon tensions and the elements of the matrix $A(q)$, called the *equilibrium* matrix, are given by

$$A_{ij} = \frac{\partial l_j}{\partial q_i}, \quad i = 1, \ldots, N, \;\; j = 1, \ldots, E. \tag{4}$$

Since the virtual work must be zero for every virtual displacement δq, we must have

$$A(q)T = 0. \tag{5}$$

At a prestresable configuration all tendons must be strictly in tension. Hence, the equilibrium equation. (5) must have positive solutions for T_j, $j = 1, \ldots, E$. The *prestressability conditions* are

$$A(q)T = 0 \quad \text{and} \quad T_j > 0 \;\; \text{for} \; j = 1, \ldots, E. \tag{6}$$

A *necessary* condition for T to have positive elements is that the kernel of $A(q)$ is nonzero. In terms of $A(q)$ this condition gives rise to the following:

$$\det(A(q)) = 0 \quad \text{if} \; N = E, \tag{7}$$

$$\det(A^{\mathrm{T}}(q)A(q)) = 0 \quad \text{if} \; N > E. \tag{8}$$

If $N < E$ the kernel of $A(q)$ is guaranteed to be nonzero.

Solving the general prestressability conditions is difficult. Previous research reported by Pellegrino (1990), Pellegrino and Calladine (1986), Motro et al. (1986), Hanaor (1988) has focused on numerical solutions. Kenner (1976) and Tarnai (1980) took an analytical approach to the prestressability problem. Tarnai (1980) presented some structures for which the determinant of the equilibrium matrix, $A(q)$, is zero for certain geometries – thus potentially leading to prestressable configurations – but did not investigate the forces in the structural members. In the following we present several tensegrity structures for which the prestressability conditions can be analytically solved for certain configurations. For these configurations the forces in the structural members are also given.

3. Two stage SVD tensegrity structures

The first example of a class of tensegrity structures for which the prestressability conditions can be analytically solved for certain prestressable configurations, is the class of two stage SVD tensegrity structures.

A perspective view of a two stage SVD tensegrity structure is given in Fig. 2. Its components are: a base, denoted by $A_{11}A_{21}A_{31}$, a top, denoted by $B_{12}B_{22}B_{32}$, three bars attached to the base through ball and socket joints ($A_{11}B_{11}, A_{21}B_{21}, A_{31}B_{31}$), three bars attached to the top through ball and socket joints ($A_{12}B_{12}$, $A_{22}B_{22}, A_{32}B_{32}$), and 18 tendons connecting the end points of the bars. The acronym SVD comes from the notation we introduce for the tendons: tendons $B_{i1}A_{j2}$ will be called *saddle* tendons, $A_{j1}B_{i1}$ and $A_{j2}B_{i2}$ *vertical* tendons, and $A_{j1}A_{i2}$ and $B_{j1}B_{i2}$ *diagonal* tendons, respectively. Points A_{ij} and B_{ij}, $i = 1, 2, 3$, $j = 1, 2$, will be called *nodal points*. For future reference a bar will be referred to by the indices of its end points (for example bar $A_{11}B_{11}$ will be the 11 bar). Stage j, where j can be equal to 1 or 2, is composed of bars ij, $i = 1, 2, 3$. We also label the tendons as follows:

$$1 = A_{11}A_{32},\quad 2 = A_{11}B_{31},\quad 3 = A_{21}A_{12},\quad 4 = A_{21}B_{11},\quad 5 = A_{31}A_{22},\quad 6 = A_{31}B_{21},\quad 7 = A_{12}B_{21},$$
$$8 = A_{12}B_{11},\quad 9 = A_{22}B_{31},\quad 10 = A_{22}B_{21},\quad 11 = A_{32}B_{11},\quad 12 = A_{32}B_{31},\quad 13 = A_{32}B_{12},$$
$$14 = B_{11}B_{12},\quad 15 = A_{12}B_{22},\quad 16 = B_{21}B_{22},\quad 17 = A_{22}B_{32},\quad 18 = B_{31}B_{32}.$$

This labeling is necessary for our formulation of the prestressability conditions: for example l_j is the length of tendon j and T_j is the tension in tendon j.

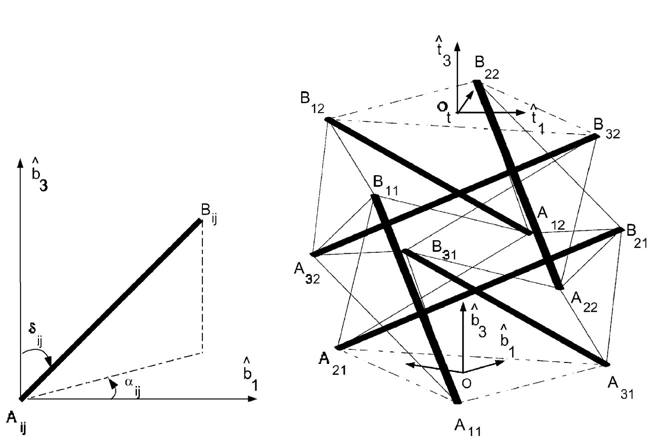

Fig. 2. Two stage SVD tensegrity structure.

The assumptions made for mathematical modeling are: the base and the top are *rigid* bodies, the bars are *rigid*, *axially symmetric*, and for each bar the rotational degree of freedom around the longitudinal axis of symmetry is neglected. This is a reasonable assumption if the points of attachment of the bars to the rigid bodies and to the tendons belong to the axes of symmetry of the bars, so that no longitudinal moment is generated, or, if we assume that the thickness of the bars is negligible. We assume that the friction forces/torques at the joints and the damping forces in the tendons are *kinetic*. We neglect the forces exerted upon the structure by external force fields (e.g. gravitational field).

3.1. Coordinate systems and generalized coordinates

The *inertial* reference frame, $\hat{b}_1$, $\hat{b}_2$, $\hat{b}_3$, is an orthonormal dextral set of vectors, whose origin coincides with the geometric center of the base triangle $A_{11}A_{21}A_{31}$. Axis $\hat{b}_3$ is orthogonal to $A_{11}A_{21}A_{31}$, pointing upward, while $\hat{b}_1$ is parallel to $A_{11}A_{31}$, pointing toward A_{31}. We introduce another dextral orthonormal reference frame, $\hat{t}_1$, $\hat{t}_2$, $\hat{t}_3$, called the *top reference frame*, which is fixed in the top rigid body. Its origin, O_t, coincides with the geometric center of the triangle $B_{12}B_{22}B_{32}$, $\hat{t}_3$ is orthogonal to $B_{12}B_{22}B_{32}$ and points upward, while $\hat{t}_1$ is parallel to $B_{12}B_{32}$ and points toward B_{32}.

The 18 independent generalized coordinates, necessary to describe the configuration of the system are listed below:

- The attitude of the top is described by the Euler angles for a 3–1–2 sequence (referred to as ψ, ϕ, θ); they characterize the rotation of the top reference frame with respect to the inertial reference frame. The inertial Cartesian coordinates, X, Y, Z, of O_t describe the translation of the top.
- For each bar ij, $i = 1, 2, 3$, $j = 1, 2$, two angles are necessary to define the inertial orientation of its axis of symmetry: the *declination* (δ_{ij}) and the *azimuth* (α_{ij}), measured with respect to $\hat{b}_1$, $\hat{b}_2$, $\hat{b}_3$ (see Fig. 2); δ_{ij} is the angle made by the vector $\overrightarrow{A_{ij}B_{ij}}$ with $\hat{b}_3$ and α_{ij} is the angle made by the projection of this vector onto plane $(\hat{b}_1, \hat{b}_2)$ with $\hat{b}_1$.

The vector of generalized coordinates is

$$q = [\delta_{11} \;\; \alpha_{11} \;\; \delta_{21} \;\; \alpha_{21} \;\; \delta_{31} \;\; \alpha_{31} \;\; \delta_{12} \;\; \alpha_{12} \;\; \delta_{22} \;\; \alpha_{22} \;\; \delta_{32} \;\; \alpha_{32} \;\; \psi \;\; \phi \;\; \theta \;\; X \;\; Y \;\; Z]^{\mathrm{T}}. \tag{9}$$

3.2. Prestressability conditions

The assumptions made for the mathematical modeling of two stage SVD tensegrity structures are particular cases of the general modeling assumptions made for the derivation of the general prestressability conditions. Thus these conditions apply here.

The inertial Cartesian coordinates of the nodal points can be easily expressed in terms of the generalized coordinates, allowing for the symbolic computation (using for example Maple) of the tendons length $l_j(q)$, $j = 1, \ldots, 18$ (see Sultan (1999) for more details).

In the case of a two stage SVD tensegrity structure the number of tendons is equal to the number of independent generalized coordinates: $E = N = 18$. The prestressability conditions are

$$A(q)T = 0 \quad \text{and} \quad T_j > 0 \quad \text{for} \;\; j = 1, \ldots, 18, \tag{10}$$

where the matrix $A(q)$ is 18×18, its elements being given by $A_{ij} = \partial l_j / \partial q_i$, $i = 1, \ldots, 18$, $j = 1, \ldots, 18$. Symbolic computational software (Maple) has been used for the derivation of A_{ij} (see Sultan (1999) for details). Since $A(q)$ is a square matrix, a *necessary* condition for prestressability is

$$\det(A(q)) = 0. \tag{11}$$

Once a solution of this equation has been found for q, further investigations must be carried out to determine if (or under what conditions) $T_j > 0$, $j = 1, \ldots, 18$.

3.3. Symmetrical prestressable configurations

In the following we introduce additional geometry specifications in order to define a class of prestressable configurations of interest of two stage SVD tensegrity structures.

We assume that all bars have equal lengths, l, and triangles $A_{11}A_{21}A_{31}$ and $B_{12}B_{22}B_{32}$ are equal equilateral triangles of edge length b. We define a *symmetrical configuration* as a configuration having the following properties: all bars have the same declination, δ, the vertical projections of points A_{i2}, B_{i1}, $i = 1, 2, 3$, onto the base make a regular hexagon, and planes $A_{11}A_{21}A_{31}$ and $B_{12}B_{22}B_{32}$ are parallel. The geometry of symmetrical configurations can be parameterized by three quantities: α, the azimuth of bar 11, δ, and h, the overlap. The overlap is defined as the distance between planes $B_{11}B_{21}B_{31}$ and $A_{12}A_{22}A_{32}$ and it is considered positive if the distance between $B_{11}B_{21}B_{31}$ and $A_{11}A_{21}A_{31}$ is greater than the distance between $A_{12}A_{22}A_{32}$ and $A_{11}A_{21}A_{31}$. The generalized coordinates corresponding to a symmetrical configuration are given by the following expressions:

$$\begin{aligned} &\psi = \frac{5\pi}{3}, \quad \theta = \phi = X = Y = 0, \quad Z = 2l\cos(\delta) - h, \\ &\alpha_{11} = \alpha_{22} = \alpha, \quad \alpha_{21} = \alpha_{32} = \alpha + \frac{4\pi}{3}, \quad \alpha_{31} = \alpha_{12} = \alpha + \frac{2\pi}{3}, \\ &\delta_{ij} = \delta, \quad i = 1,2,3, \ \ j = 1,2. \end{aligned} \tag{12}$$

Top and frontal views of a two stage SVD tensegrity structure with this geometry are given in Fig. 3.

Simple geometrical considerations show that in such a configuration, all of the saddle, vertical, and diagonal tendons have the same lengths,

$$S = \sqrt{h^2 + \frac{b^2}{3} + l^2\sin^2(\delta) - \frac{2}{\sqrt{3}} lb\sin(\delta)\cos\left(\alpha - \tfrac{\pi}{6}\right)}, \tag{13}$$

$$V = \sqrt{b^2 + l^2 - 2lb\sin(\delta)\sin\left(\alpha + \tfrac{\pi}{6}\right)}, \tag{14}$$

$$D = \sqrt{h^2 + \frac{b^2}{3} + l^2 - \frac{2}{\sqrt{3}} lb\sin(\delta)\sin(\alpha) - 2lh\cos(\delta)}, \tag{15}$$

respectively (see Sultan (1999) for details). Here S represents the length of a saddle tendon, V represents the length of a vertical tendon, and D represents the length of a diagonal tendon.

Prestressable equilibria satisfying the above geometrical specifications will be called *symmetrical prestressable configurations*.

We next define the range of α and δ. We remark that for $\alpha = \frac{\pi}{6}$ a symmetrical configuration cannot exist because the axes of symmetry of bars 11, 21, 31 intersect. Thus $\alpha \in \{[0, 2\pi) - \frac{\pi}{6}\}$. It is easy to see that, from mechanical equilibrium considerations, for $\delta = 0$ (or $\delta = \pi$) the structure cannot be in a symmetrical prestressable configuration. Also, for $\delta = \frac{\pi}{2}$ a symmetrical prestressable configuration is not physically feasible because, from mechanical equilibrium considerations, planes $A_{11}A_{21}A_{31}$ and $B_{12}B_{22}B_{32}$ should coincide, a situation we do not consider here. Thus we restrict δ by

$$0 < \delta < \frac{\pi}{2}. \tag{16}$$

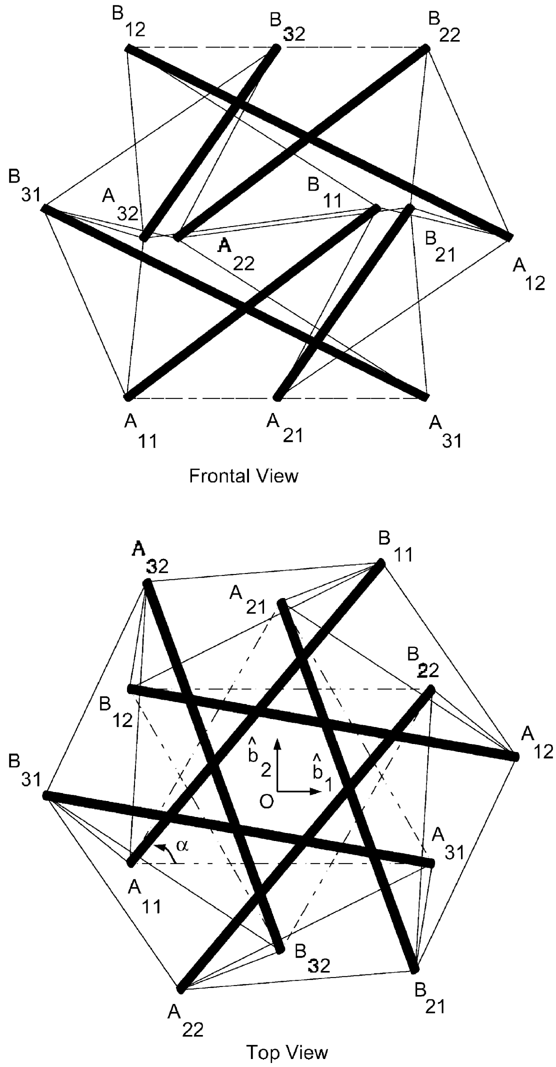

Fig. 3. Symmetrical configuration.

The necessary prestressability condition for symmetrical prestressable configurations has been derived by substituting Eq. (12) in Eq. (11) using Maple. The resulting equation,

$$\det(A(\alpha,\delta,h)) = 0, \tag{17}$$

and the prestressability conditions,

$$A(\alpha,\delta,h)T = 0, \quad T_j > 0 \quad \text{for} \quad j = 1,\ldots,18, \tag{18}$$

have been numerically investigated using Matlab as follows: for fixed l, b, α, and δ, a solution of Eq. (17) for h has been sought. Once such a solution has been found, equation $A(\alpha,\delta,h)T = 0$ has been solved for T. Numerous numerical experiments have indicated that there is atmost one solution of Eq. (17) for h with $T_j > 0,\ j = 1,\ldots,18$, and that at such a solution the tensions in the saddle, vertical, and diagonal tendons are, respectively, equal (see Sultan (1999) for details). These numerical results suggest that we impose the condition that the tensions in the saddle, vertical, and diagonal tendons are equal to T_S, T_V, and T_D, respectively, so that the structure of the vector T is

$$T = [T_D \ T_V \ T_D \ T_V \ T_D \ T_V \ T_S \ T_S \ T_S \ T_S \ T_S \ T_S \ T_V \ T_D \ T_V \ T_D \ T_V \ T_D]^T. \tag{19}$$

In other words, within the class of symmetrical prestressable configurations, we search for those configurations for which the tensions in the saddle, vertical, and diagonal tendons are, respectively, equal.

Substitution of Eqs. (12) and (19) into the prestressability conditions (10) leads to

$$A_r T_r = 0, \quad T_S > 0, \quad T_V > 0, \quad T_D > 0, \tag{20}$$

where the matrix A_r is 18×3 and $T_r = [T_S \ T_V \ T_D]^T$. Symbolic computational software (Maple) has been used to derive Eq. (20). By inspection of A_r we have ascertained that only three rows of A_r can be independent: 1, 2, and 18 (see Sultan (1999) for details on A_r). The corresponding prestressability equations and the inequalities on the tensions yield

$$A_e T_r = 0, \quad T_S > 0, \quad T_V > 0, \quad T_D > 0. \tag{21}$$

Here

$$A_e = \begin{bmatrix} \frac{b\sin\left(\alpha - \frac{\pi}{6}\right)}{\sqrt{3}S} & -\frac{b\cos\left(\alpha + \frac{\pi}{6}\right)}{V} & -\frac{b\cos(\alpha)}{\sqrt{3}D} \\ \frac{l\sin(2\delta)}{2S} - \frac{b\cos(\delta)\cos\left(\alpha - \frac{\pi}{6}\right)}{\sqrt{3}S} & -\frac{b\cos(\delta)\sin\left(\alpha + \frac{\pi}{6}\right)}{V} & \frac{\sqrt{3}h\sin(\delta) - b\sin(\alpha)\cos(\delta)}{\sqrt{3}D} \\ \frac{h}{S} & 0 & \frac{h - l\cos(\delta)}{D} \end{bmatrix}. \tag{22}$$

We now give an important result of this paper.

Theorem 1. *For a given $\alpha \in \{[0, 2\pi) - \frac{\pi}{6}\}$ and a given $\delta \in (0, \frac{\pi}{2})$, the two stage SVD tensegrity structure might yield atmost one symmetrical prestressable configuration for which the tensions in the saddle, vertical, and diagonal tendons are respectively equal. Necessary and sufficient conditions for such a configuration to occur are given by*

$$\frac{\pi}{6} < \alpha < \frac{\pi}{2}, \quad 0 < \delta < \frac{\pi}{2},$$

$$l\sin(\delta)\left|\cos\left(\alpha + \tfrac{\pi}{6}\right)\right| < \frac{b}{2\sqrt{3}} \quad \text{and} \quad \sin\left(\alpha + \tfrac{\pi}{6}\right) < \frac{3l\sin(\delta)}{2b}. \tag{23}$$

The value of the corresponding overlap at such a configuration is given by

$$h = \begin{cases} \frac{\cos(\delta)}{2u}\left(lu + \sqrt{\frac{b^2}{3} - 3l^2u^2} - \frac{b}{\sqrt{3}}\right) & \text{if } \alpha \neq \frac{\pi}{3}, \\ \frac{l\cos(\delta)}{2} & \text{if } \alpha = \frac{\pi}{3}, \end{cases} \tag{24}$$

where

$$u = \sin(\delta)\cos\left(\alpha + \tfrac{\pi}{6}\right). \tag{25}$$

In addition, the tensions in the tendons are given by Eq. (19) *with*

$$T_r = [T_S \ T_V \ T_D]^T = PT_0, \tag{26}$$

where P is an arbitrary positive scalar called the pretension coefficient and T_0 is given by

$$T_0^{\mathrm{T}} = [T_{0_{\mathrm{S}}}\ T_{0_{\mathrm{V}}}\ T_{0_{\mathrm{D}}}] = \frac{[T_{\mathrm{S}}^{\mathrm{r}}\ T_{\mathrm{V}}^{\mathrm{r}}\ T_{\mathrm{D}}^{\mathrm{r}}]}{\sqrt{6}\ \|\ [T_{\mathrm{S}}^{\mathrm{r}}\ T_{\mathrm{V}}^{\mathrm{r}}\ T_{\mathrm{D}}^{\mathrm{r}}]\ \|}, \tag{27}$$

where

$$T_{\mathrm{V}}^{\mathrm{r}} = \begin{cases} \frac{V}{D}\frac{1}{\sqrt{3}\cos\left(\alpha+\frac{\pi}{6}\right)}\left(\left(\frac{l\cos(\delta)}{h}-1\right)\sin\left(\alpha-\frac{\pi}{6}\right)-\cos(\alpha)\right)T_{\mathrm{D}}^{\mathrm{r}} & \mathit{if}\ \alpha \neq \frac{\pi}{3}, \\ \frac{VT_{\mathrm{D}}^{\mathrm{r}}}{D}\left(\frac{3l}{2b}\sin(\delta)-1\right) & \mathit{if}\ \alpha = \frac{\pi}{3}, \end{cases} \tag{28}$$

$$T_{\mathrm{S}}^{\mathrm{r}} = \begin{cases} \frac{S}{D}\left(\frac{l\cos(\delta)}{h}-1\right)T_{\mathrm{D}}^{\mathrm{r}} & \mathit{if}\ \alpha \neq \frac{\pi}{3}, \\ T_{\mathrm{D}}^{\mathrm{r}} & \mathit{if}\ \alpha = \frac{\pi}{3}, \end{cases} \tag{29}$$

$$T_{\mathrm{D}}^{\mathrm{r}} = 1. \tag{30}$$

Proof. See Appendix A. □

We note that numerous numerical experiments indicated that these prestressable configurations are stable. This conclusion was reached by investigating the linearized dynamical models around these particular equilibrium solutions. The stiffness matrix of each of the linearized models was positive definite (see Sultan (1999) for details).

Assuming that the tendons are linear elastic, the tendon rest lengths which are necessary to assure a desired symmetrical prestressable configuration (characterized by α and δ) for which the tensions in the saddle, vertical, and diagonal tendons are respectively equal, and a prescribed pretension, P, can be derived as follows. The tendons are linear elastic, thus

$$T_j = k_j\frac{l_j - l_{0_j}}{l_{0_j}}, \quad j = 1,\ldots,18. \tag{31}$$

Here T_j, k_j, l_j, l_{0_j} denote the tension, stiffness, length, and rest length of the jth tendon respectively. If we take into account that $T = PT_0$ and solve for the rest lengths we get

$$l_{0_j} = \frac{k_j l_j}{T_{0_j}P + k_j}, \quad j = 1,\ldots,18. \tag{32}$$

Equilibrium conditions of the nodal points A_{j2} or B_{j1}, $j = 1,2,3$, provide the forces in the bars. These are all axial forces, are equal, and depend linearly on P; they are given by

$$C = C_0 P, \tag{33}$$

where

$$\begin{aligned} C_0 = \frac{T_{0_{\mathrm{D}}}}{6Dh\sin\left(\frac{\pi}{3}-\alpha\right)}\Bigg(&2\sqrt{3}hb\sin(\delta) - \frac{\sqrt{3}}{2}lb\sin(2\delta) + 6h^2\cos(\delta)\sin\left(\alpha-\frac{\pi}{3}\right) \\ &- 6lh\left(\cos^2(\delta)\sin\left(\alpha-\frac{\pi}{3}\right) - \frac{1}{\sqrt{3}}\sin\left(\alpha+\frac{\pi}{6}\right)\right) + 2\sqrt{3}l^2\cos(\delta)\cos(\alpha) \\ &+ 6l^2\cos^3(\delta)\sin\left(\alpha-\frac{\pi}{3}\right)\Bigg) \quad \text{if } \alpha \neq \frac{\pi}{3} \end{aligned} \tag{34}$$

and

$$C_0 = \frac{T_{0_D}}{D}\left(\frac{3l^2}{2b}\sin(\delta) - \frac{l}{2}\right) \quad \text{if } \alpha = \frac{\pi}{3}. \tag{35}$$

A positive C_0 means that the bars are compressed.

3.4. Two stage SD tensegrity structures

Another class of tensegrity structures for which the prestressability conditions can be analytically solved for certain prestressable configurations, is that of two stage SD tensegrity structures. This can be easily proved from the two stage SVD tensegrity structures analysis presented before.

A two stage SD tensegrity structure can be obtained from a two stage SVD structure by eliminating the vertical tendons. We ascertain from the SVD tensegrity structures analysis that, in a two stage SVD tensegrity structure yielding symmetrical prestressable configurations for which the saddle and diagonal tendon tensions are respectively equal, the tensions in the vertical tendons can be zero if

$$\sin\left(\alpha + \tfrac{\pi}{6}\right) = \frac{3l\sin(\delta)}{2b}. \tag{36}$$

Hence we conclude that a two stage SD tensegrity structure can yield symmetrical prestressable configurations for which the saddle and diagonal tendon tensions are respectively equal and that some of these prestressable configurations are limit cases of the symmetrical prestressable configurations of two stage SVD tensegrity structures for which the saddle, vertical, and diagonal tendon tensions are respectively equal.

It can also be proved that actually *all* of the prestressable configurations of two stage SD tensegrity structures for which the tensions in the diagonal and saddle tendons are respectively equal are limit cases of the SVD ones. The proof is summarized in the following (a detailed proof can be found in Sultan (1999)).

Let T_S denote the tension in all saddle tendons and T_D denote the tension in all diagonal tendons. Going through the same procedure as in the SVD case, we obtain the prestressability conditions in the form

$$[A_1 \ \ A_3]T_{SD} = 0, \quad T_S > 0, \ \ T_D > 0, \tag{37}$$

A_1 and A_3 being the first and third columns of A_e and $T_{SD} = [T_S \ \ T_D]^T$. For nonzero solutions A_1 and A_3 must be linearly dependent, a condition which leads to Eq. (36) and

$$h = -\frac{\cos\left(\alpha + \frac{\pi}{3}\right)}{3\cos\left(\alpha - \frac{\pi}{3}\right)}\sqrt{9l^2 - 4b^2\cos^2\left(\alpha - \frac{\pi}{3}\right)} = -l\frac{\cos\left(\alpha + \frac{\pi}{3}\right)}{\cos\left(\alpha - \frac{\pi}{3}\right)}\cos(\delta). \tag{38}$$

Eq. (36) is equivalent, as expected, with the condition $T_V = 0$ deduced for the symmetrical prestressable configurations for which the saddle, vertical, and diagonal tendon tensions are respectively equal of two stage SVD tensegrity structures. Furthermore, Eq. (38) can be obtained as a particular case of Eq. (24) if we substitute Eq. (36) in the solution for the overlap for the SVD type.

The constraint on the overlap ($0 < h < l\cos(\delta)$) holds for these prestressable configurations of the SD tensegrity type by the same argument as for the SVD type. Constraints $0 < h < l\cos(\delta)$, $0 < \delta < \frac{\pi}{2}$, and Eq. (36) lead to the following necessary and sufficient conditions for a symmetrical prestressable configuration for which the tensions in the saddle and diagonal tendons are respectively equal, to exist:

$$\frac{\pi}{6} < \alpha < \frac{\pi}{2} \quad \text{and} \quad \frac{2b}{3l}\sin\left(\alpha + \tfrac{\pi}{6}\right) < 1. \tag{39}$$

For each α verifying Eq. (39), the corresponding h is given by Eq. (38), while δ is given by Eq. (36).

The formulas describing the forces in the tendons and bars can be obtained from the ones corresponding to two stage SVD tensegrity structures by enforcing the condition that the tension in the vertical tendons is 0 (which is equivalent to Eq. (36)).

3.5. Prestressable equilibrium surface

For a two stage SVD tensegrity structure yielding symmetrical prestressable configurations for which the saddle, vertical, and diagonal tendon tensions are respectively equal, the set of all h, α, δ must satisfy Eqs. (23) and (24). In the three dimensional space of α, δ, h this set represents a surface which we call the *prestressable equilibrium surface*. A particular subset of this set, corresponding to all nodal points lying on the surface of a cylinder, is defined by the *cylindrical constraint*:

$$\sin(\delta) = \frac{2b\sin\left(\alpha + \frac{\pi}{3}\right)}{\sqrt{3}l}. \tag{40}$$

Another particular subset, corresponding to all nodal points lying on the surface of a sphere, is defined by the *spherical constraint*:

$$h = \frac{\sqrt{3}l\cos(2\delta) + 2b\sin(\delta)\cos\left(\alpha - \frac{\pi}{6}\right)}{\sqrt{3}\cos(\delta)}. \tag{41}$$

Figs. 4 and 5 show the prestressable equilibrium surface, obtained for the following geometrical parameters:

$$l = 0.4 \text{ m}, \quad b = 0.27 \text{ m}. \tag{42}$$

The state of stress of a two stage SVD tensegrity structure yielding symmetrical prestressable configurations for which the saddle, vertical, and diagonal tendon tensions are respectively equal, is characterized by the forces in its members. Figs. 6–9 represent the contour plots of the basis compression (C_0) and tensions ($T_{0_S}, T_{0_V}, T_{0_D}$) respectively. The numbers on the level curves of these plots represent the values of the quantity whose variation with α and δ is represented (C_0, T_{0_S}, T_{0_V}, and T_{0_D} respectively).

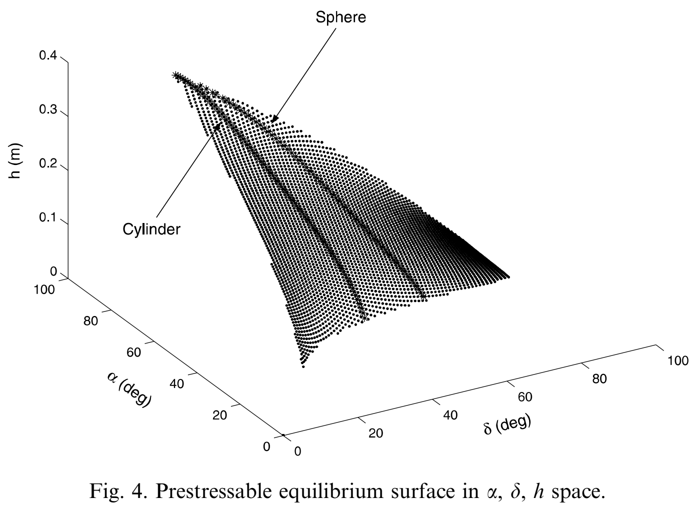

Fig. 4. Prestressable equilibrium surface in α, δ, h space.

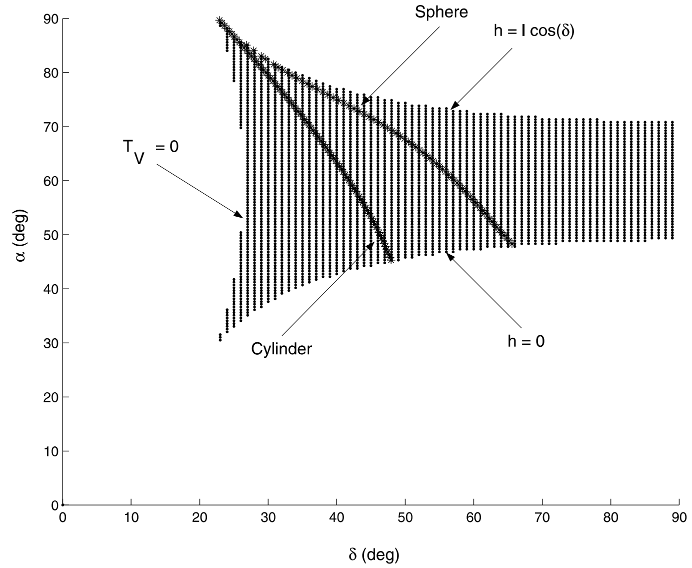

Fig. 5. Prestressable equilibrium surface projection onto α, δ plane.

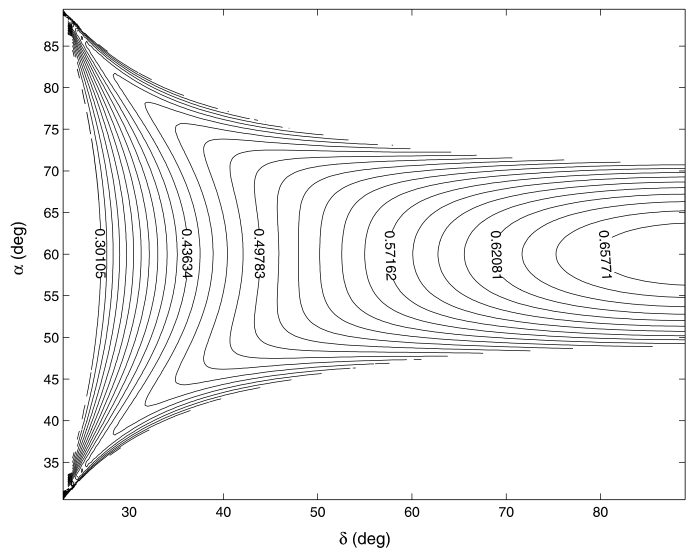

Fig. 6. Compressive force (C_0) variation.

4. Two stage SVDB tensegrity structures

The next example of a class of tensegrity structures for which the prestressability conditions can be analytically solved for certain prestressable configurations, is that of two stage SVDB tensegrity structures.

A two stage SVDB tensegrity structure can be obtained from a two stage SVD structure if we replace the top and bottom rigid bodies with tendons connecting the nodal points A_{11}, A_{21}, A_{31}, and B_{11}, B_{21}, B_{31},

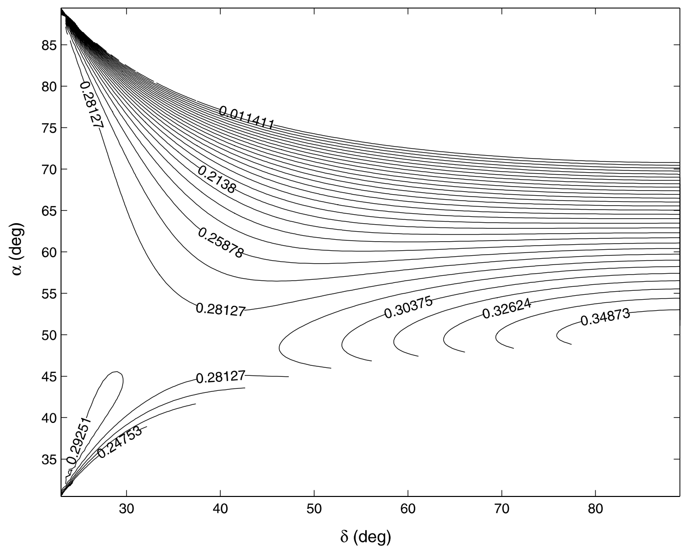

Fig. 7. Saddle tension (T_{0_s}) variation.

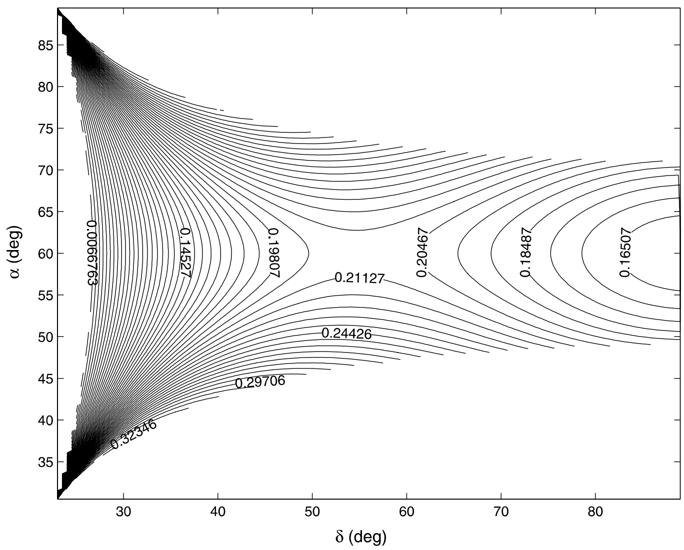

Fig. 8. Vertical tension (T_{0_v}) variation.

respectively. Thus a two stage SVDB tensegrity structure is composed of six bars and 24 tendons (see Fig. 1). We shall use the same notation as for the two stage SVD type (e.g. tendons $B_{i1}A_{j2}$ will be called saddle tendons, $A_{j1}B_{i1}$ and $A_{j2}B_{i2}$ vertical tendons, and $A_{j1}A_{i2}$ and $B_{j1}B_{i2}$ diagonal tendons); the tendons which replace the top and bottom rigid bodies – $A_{11}A_{21}$, $A_{21}A_{31}$, $A_{31}A_{11}$, $B_{12}B_{22}$, $B_{22}B_{32}$, $B_{32}B_{12}$ – will be called *boundary tendons*. For prestressability conditions derivation we label the tendons as follows:

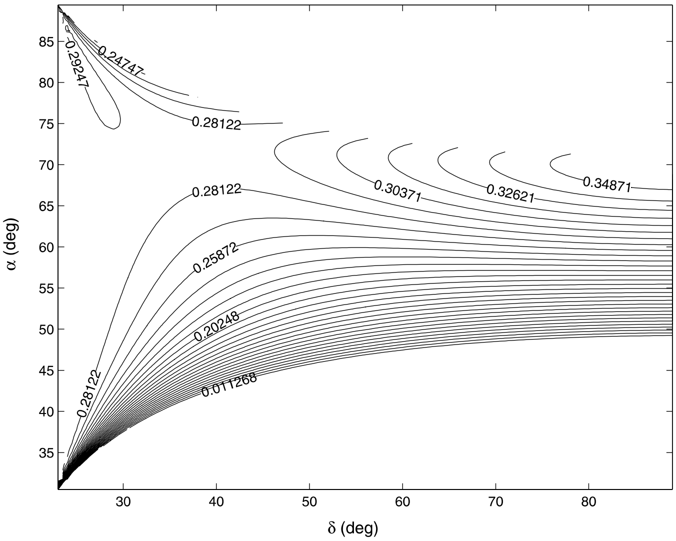

Fig. 9. Diagonal tension (T_{0_D}) variation.

$$1 = A_{11}A_{21},\quad 2 = A_{21}A_{31},\quad 3 = A_{31}A_{11},\quad 4 = A_{11}A_{32},\quad 5 = A_{11}B_{31},\quad 6 = A_{21}A_{12},\quad 7 = A_{21}B_{11},$$
$$8 = A_{31}A_{22},\quad 9 = A_{31}B_{21},\quad 10 = A_{12}B_{21},\quad 11 = A_{12}B_{11},\quad 12 = A_{22}B_{31},\quad 13 = A_{22}B_{21},$$
$$14 = A_{32}B_{11},\quad 15 = A_{32}B_{31},\quad 16 = A_{32}B_{12},\quad 17 = B_{11}B_{12},\quad 18 = A_{12}B_{22},\quad 19 = B_{21}B_{22},$$
$$20 = A_{22}B_{32},\quad 21 = B_{31}B_{32},\quad 22 = B_{12}B_{22},\quad 23 = B_{22}B_{32},\quad 24 = B_{32}B_{12}.$$

We also make the same modeling assumptions as for two stage SVD tensegrity structures.

The independent generalized coordinates used to describe the configuration of this system with respect to an inertial, dextral, orthonormal reference frame, $\hat{b}_1$, $\hat{b}_2$, $\hat{b}_3$, are the Cartesian coordinates, x_{ij}, y_{ij}, z_{ij}, of the mass center of bar ij, $i = 1, 2, 3$, $j = 1, 2$, with respect to the inertial reference frame, and the azimuth α_{ij} and the declination δ_{ij}, which characterize the orientation of bar ij with respect to the inertial frame (these angles are defined in the same way as the declination and azimuth used for a two stage SVD structure). The number of independent generalized coordinates is $N = 30$ and the vector of generalized coordinates is:

$$q = [x_{11}\ y_{11}\ z_{11}\ \delta_{11}\ \alpha_{11}\ x_{21}\ y_{21}\ z_{21}\ \delta_{21}\ \alpha_{21}\ x_{31}\ y_{31}\ z_{31}\ \delta_{31}\ \alpha_{31}\ x_{12}\ y_{12}\ z_{12}\ \delta_{12}\ \alpha_{12}\ x_{22}\ y_{22}\ z_{22}\ \delta_{22}\ \alpha_{22}\ x_{32}\ y_{32}\ z_{32}\ \delta_{32}\ \alpha_{32}]^{\mathrm{T}}. \tag{43}$$

4.1. Symmetrical prestressable configurations

The Cartesian coordinates of the nodal points with respect to the inertial reference frame can be easily expressed in terms of the generalized coordinates. Using these coordinates the lengths of the tendons, $l_j(q)$, $j = 1, \ldots, 24$, and the elements of the matrix $A(q)$, $A_{ij} = \partial l_j / \partial q_i$, $i = 1, \ldots, 30$, $j = 1, \ldots, 24$, can be derived using symbolic computation (Maple) (see Sultan (1999) for details).

For two stage SVDB tensegrity structures, the prestressability conditions become

$$A(q)T = 0 \quad \text{and} \quad T_j > 0 \ \text{for}\ j = 1, \ldots, 24 \tag{44}$$

where $A(q)$ is a 30×24 matrix. In this case the equilibrium matrix $A(q)$ is not square, the number of generalized coordinates ($N = 30$) being greater than the number of tendons ($E = 24$). Thus a necessary condition for prestressability is $\det(A^{\mathrm{T}}(q)A(q)) = 0$.

As for two stage SVD tensegrity structures, we shall consider the symmetrical prestressable configurations class. Recall that the symmetrical configurations class has the following properties: all bars have equal lengths (l), triangles $A_{11}A_{21}A_{31}$ and $B_{12}B_{22}B_{32}$ are equal, parallel, equilateral triangles of side b, all bars have the same declination, δ, the vertical projections of points A_{i2}, B_{i1}, $i = 1, 2, 3$, onto the plane $A_{11}A_{21}A_{31}$ make a regular hexagon. This geometry can be parameterized in terms of α, the azimuth of bar 11, δ, and h, the overlap (the distance between planes $A_{12}A_{22}A_{32}$ and $B_{11}B_{21}B_{31}$ considered positive if $A_{12}A_{22}A_{32}$ is closer to $A_{11}A_{21}A_{31}$ than $B_{11}B_{21}B_{31}$). Consider that the *inertial* reference frame, $\hat{b}_1, \hat{b}_2, \hat{b}_3$, is an orthonormal dextral set of vectors, which at equilibrium is located at the geometric center of the triangle $A_{11}A_{21}A_{31}$ and that $\hat{b}_3$ is orthogonal to $A_{11}A_{21}A_{31}$ pointing upward, while $\hat{b}_1$ is parallel to $A_{11}A_{31}$ pointing toward A_{31}. Then the corresponding generalized coordinates have the following values:

$$\begin{aligned}
&x_{11} = \frac{l}{2}\sin(\delta)\cos(\alpha) - \frac{b}{2}, \quad y_{11} = \frac{l}{2}\sin(\delta)\sin(\alpha) - \frac{b}{2\sqrt{3}}, \quad z_{11} = l\cos(\delta) - h, \quad \alpha_{11} = \alpha,\\
&x_{21} = \frac{l}{2}\sin(\delta)\cos(\alpha_{21}), \quad y_{21} = \frac{b}{\sqrt{3}} + \frac{l}{2}\sin(\delta)\sin(\alpha_{21}), \quad z_{21} = l\cos(\delta) - h, \quad \alpha_{21} = \alpha + \frac{4\pi}{3},\\
&x_{31} = \frac{b}{2} + \frac{l}{2}\sin(\delta)\cos(\alpha_{31}), \quad y_{31} = \frac{l}{2}\sin(\delta)\sin(\alpha_{31}) - \frac{b}{2\sqrt{3}}, \quad z_{31} = l\cos(\delta) - h,\\
&\alpha_{31} = \alpha + \frac{2\pi}{3}, \quad x_{12} = \frac{l}{4}\sin(\delta)\cos(\alpha) + \frac{\sqrt{3}}{4}l\sin(\delta)\sin(\alpha) - \frac{b}{2},\\
&y_{12} = \frac{b}{2\sqrt{3}} - \frac{\sqrt{3}}{4}l\sin(\delta)\cos(\alpha) + \frac{l}{4}\sin(\delta)\sin(\alpha), \quad z_{12} = \frac{3}{2}l\cos(\delta) - h, \alpha_{12} = \alpha + \frac{2\pi}{3},\\
&x_{22} = \frac{b}{2} - \frac{l}{2}\sin(\delta)\cos(\alpha), \quad y_{22} = \frac{b}{2\sqrt{3}} - \frac{l}{2}\sin(\delta)\sin(\alpha), \quad z_{22} = \frac{3}{2}l\cos(\delta) - h, \quad \alpha_{22} = \alpha,\\
&x_{32} = \frac{l}{4}\sin(\delta)\cos(\alpha) - \frac{\sqrt{3}}{4}l\sin(\delta)\sin(\alpha),\\
&y_{32} = \frac{l}{4}\sin(\delta)\sin(\alpha) + \frac{\sqrt{3}}{4}l\sin(\delta)\cos(\alpha) - \frac{b}{\sqrt{3}},\\
&z_{32} = \frac{3}{2}l\cos(\delta) - h, \quad \alpha_{32} = \alpha + \frac{4\pi}{3}, \quad \delta_{ij} = \delta, \quad i = 1, 2, 3, \ \ j = 1, 2.
\end{aligned} \tag{45}$$

At a symmetrical configuration all saddle, vertical, and diagonal tendons have the same lengths, respectively, given by the same formulas (13)–(15) as for the SVD type. The boundary tendons have the same length, b.

As for the SVD case, in a symmetrical prestressable configuration, α and δ are restricted to $\alpha \in \{[0, 2\pi) - \frac{\pi}{6}\}$ and $\delta \in (0, \frac{\pi}{2})$.

The corresponding prestressability conditions, $A(\alpha, \delta, h)T = 0$, $T_j > 0$, $j = 1, \dots, 24$, have been numerically investigated as follows. For fixed l, b, α, and δ the *necessary* prestressability condition, $\det(A(\alpha, \delta, h)^{\mathrm{T}}A(\alpha, \delta, h)) = 0$, has been solved for h and the kernel of the corresponding equilibrium matrix has been computed by solving $A(\alpha, \delta, h)T = 0$ for T. We ascertained that the prestressability conditions have at most one solution for h such that $T_j > 0$ for $j = 1, \dots, 24$, and that, in this case, the tensions in the saddle, vertical, diagonal, and boundary tendons are respectively equal. Thus we impose the condition that the tensions in the saddle, vertical, diagonal, and boundary tendons are respectively equal to T_{S}, T_{V}, T_{D}, and T_{B} such that the vector of tensions has the following structure:

$$T = [T_{\mathrm{B}}\ T_{\mathrm{B}}\ T_{\mathrm{B}}\ T_{\mathrm{D}}\ T_{\mathrm{V}}\ T_{\mathrm{D}}\ T_{\mathrm{V}}\ T_{\mathrm{D}}\ T_{\mathrm{V}}\ T_{\mathrm{S}}\ T_{\mathrm{S}}\ T_{\mathrm{S}}\ T_{\mathrm{S}}\ T_{\mathrm{S}}\ T_{\mathrm{S}}\ T_{\mathrm{V}}\ T_{\mathrm{D}}\ T_{\mathrm{V}}\ T_{\mathrm{D}}\ T_{\mathrm{V}}\ T_{\mathrm{D}}\ T_{\mathrm{B}}\ T_{\mathrm{B}}\ T_{\mathrm{B}}]^{\mathrm{T}}. \tag{46}$$

If we substitute Eqs. (45) and (46) into Eq. (44) we obtain the reduced prestressability conditions

$$A_r T_r = 0, \quad T_S > 0, \quad T_V > 0, \quad T_D > 0, \quad T_B > 0, \tag{47}$$

where A_r is a 30×4 matrix, and $T_r = [T_S \ T_V \ T_D \ T_B]^T$.

Further investigations showed that the only rows of A_r which might be independent are 1, 2, 3, and 4 (see Sultan (1999) for details). The prestressability conditions corresponding to these rows of A_r yield

$$A_e T_r = 0, \quad T_S > 0, \quad T_V > 0, \quad T_D > 0, \quad T_B > 0, \tag{48}$$

where the structure of A_e is

$$A_e = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & 0 & * & 0 \\ * & * & * & * \end{bmatrix} \tag{49}$$

with the nonzero elements of A_e given by

$$\begin{aligned}
&A_{e_{11}} = \frac{1}{S}(2l\sin(\delta)\cos(\alpha) - b), \qquad A_{e_{12}} = \frac{1}{V}\left(2\sqrt{3}l\sin(\delta)\cos\left(\alpha - \frac{\pi}{6}\right) - 3b\right),\\
&A_{e_{13}} = \frac{1}{D}\left(2l\sin(\delta)\cos\left(\alpha - \frac{\pi}{3}\right) - b\right), \qquad A_{e_{14}} = -3, \qquad A_{e_{21}} = \frac{1}{S}\left(2l\sin(\delta)\sin(\alpha) - \frac{b}{\sqrt{3}}\right),\\
&A_{e_{22}} = \frac{\sqrt{3}}{V}(2l\sin(\delta)\sin\left(\alpha - \frac{\pi}{6}\right) - b), \qquad A_{e23} = \frac{1}{D}\left(2l\sin(\delta)\sin\left(\alpha - \frac{\pi}{3}\right) - \frac{b}{\sqrt{3}}\right),\\
&A_{e_{24}} = -\sqrt{3}, \qquad A_{e_{31}} = \frac{4h}{S}, \qquad A_{e33} = \frac{4}{D}(h - l\cos(\delta)),\\
&A_{e_{41}} = \frac{1}{S}(l\sin(\delta)\cos(\delta) - \frac{b}{\sqrt{3}}\cos(\delta)\sin\left(\alpha + \frac{\pi}{3}\right) - 2h\sin(\delta)),\\
&A_{e_{42}} = \frac{\cos(\delta)}{V}\left(b\cos\left(\alpha + \frac{\pi}{3}\right) - \frac{3}{2}l\sin(\delta)\right), \qquad A_{e_{43}} = \frac{\cos(\delta)}{D}\left(b\cos\left(\alpha + \frac{\pi}{3}\right) + \frac{3}{2}l\sin(\delta)\right),\\
&A_{e_{44}} = \sqrt{3}\cos(\delta)\cos\left(\alpha - \frac{\pi}{6}\right).
\end{aligned} \tag{50}$$

We now give another important result of this paper.

Theorem 2. *For $\alpha \in \{[0, 2\pi) - \frac{\pi}{6}\}$ and $\delta \in (0, \frac{\pi}{2})$ the set of symmetrical prestressable configurations of two stage SVDB tensegrity structures for which the tensions in the saddle, vertical, diagonal, and boundary tendons are respectively equal is identical with the set of symmetrical prestressable configurations of two stage SVD tensegrity structures for which the tensions in the saddle, vertical, and diagonal tendons are respectively equal.*

Proof. See Appendix B. □

Finally we note that the forces acting on the bars are all equal and their expression, in terms of α, δ, and P, is given by Eqs. (33)–(35).

4.2. Two stage SDB tensegrity structures

Another example of a class of tensegrity structures for which the prestressability conditions have analytical solutions is that of two stage SDB tensegrity structures.

A two stage SDB tensegrity structure can be obtained by removing the vertical tendons from a two stage SVDB one. In the same way as for the SD type, we ascertain that two stage SDB tensegrity structures can yield symmetrical prestressable configurations for which the saddle and diagonal tendon tensions are respectively equal and that the necessary condition for this to happen is given by Eq. (36) (which is equivalent to $T_V = 0$).

In Sultan (1999) it has been proved that all of the symmetrical prestressable configurations for which the saddle, diagonal, and boundary tendon tensions are respectively equal of two stage SDB tensegrity structures are limit cases of the symmetrical prestressable configurations for which the saddle, vertical, diagonal, and boundary tendon tensions are respectively equal of two stage SVDB tensegrity structures. The necessary formulas (describing the equilibrium solutions and the state of stress) are obtained from the ones corresponding to the two stage SVDB type by enforcing the condition that the tensions in the vertical tendons are 0.

5. Two stage SVDT tensegrity structures

The next example of a class of tensegrity structures for which analytical solutions of the prestressability conditions have been found is that of two stage SVDT tensegrity structures.

A two stage SVDT tensegrity structure is obtained from a SVD one if we replace the top rigid body with tendons connecting the nodal points labeled as B_{11}, B_{21}, and B_{31}. A SVDT structure is composed of six bars and 21 tendons. We shall use the same notation as for the two stage SVD type (e.g. tendons $B_{i1}A_{j2}$ will be called saddle tendons, $A_{j1}B_{i1}$ and $A_{j2}B_{i2}$ vertical tendons, and $A_{j1}A_{i2}$ and $B_{j1}B_{i2}$ diagonal tendons); the tendons which replace the top rigid body – $B_{12}B_{22}$, $B_{22}B_{32}$, $B_{32}B_{12}$ – will be called *top* tendons. The tendons are labeled as follows:

$$1 = A_{11}A_{32},\quad 2 = A_{11}B_{31},\quad 3 = A_{21}A_{12},\quad 4 = A_{21}B_{11},\quad 5 = A_{31}A_{22},\quad 6 = A_{31}B_{21},\quad 7 = A_{12}B_{21},$$
$$8 = A_{12}B_{11},\quad 9 = A_{22}B_{31},\quad 10 = A_{22}B_{21},\quad 11 = A_{32}B_{11},\quad 12 = A_{32}B_{31},\quad 13 = A_{32}B_{12},$$
$$14 = B_{11}B_{12},\quad 15 = A_{12}B_{22},\quad 16 = B_{21}B_{22},\quad 17 = A_{22}B_{32},\quad 18 = B_{31}B_{32},\quad 19 = B_{12}B_{22},$$
$$20 = B_{22}B_{32},\quad 21 = B_{32}B_{12}.$$

We make the same modeling assumptions as for the two stage SVD tensegrity structures and we define the inertial reference frame in the same way as the one used for the analysis of two stage SVD tensegrity structures: it is a dextral orthonormal set of vectors, $\hat{b}_1$, $\hat{b}_2$, $\hat{b}_3$, with origin at the geometric center of the base triangle $A_{11}A_{21}A_{31}$, axis $\hat{b}_3$ is orthogonal to $A_{11}A_{21}A_{31}$, pointing upward, while $\hat{b}_1$ is parallel to $A_{11}A_{31}$, pointing toward A_{31}.

The independent generalized coordinates used to describe the configuration of this system with respect to the inertial reference frame are the Cartesian coordinates, x_{i2}, y_{i2}, z_{i2}, of the mass center of bar $i2$, $i = 1, 2, 3$, with respect to the inertial reference frame, and the azimuth α_{ij} and the declination δ_{ij}, which characterize the orientation of bar ij, $i = 1, 2, 3$, $j = 1, 2$ with respect to the inertial frame (these angles are defined in the same way as the declination and azimuth used for two stage SVD tensegrity structures). The number of independent generalized coordinates is $N = 21$ and the vector of generalized coordinates is:

$$q = [\delta_{11}\ \alpha_{11}\ \delta_{21}\ \alpha_{21}\ \delta_{31}\ \alpha_{31}\ x_{12}\ y_{12}\ z_{12}\ \delta_{12}\ \alpha_{12}\ x_{22}\ y_{22}\ z_{22}\ \delta_{22}\ \alpha_{22}\ x_{32}\ y_{32}\ z_{32}\ \delta_{32}\ \alpha_{32}]^{\mathrm{T}}. \tag{51}$$

5.1. Symmetrical prestressable configurations

As for the SVD and SVDB types the inertial Cartesian coordinates of the nodal points, the lengths of the tendons $l_j(q)$, $j = 1, \ldots, 21$, and the elements of the equilibrium matrix $A(q)$, $A_{ij} = \partial l_j/\partial q_i$, $i = 1, \ldots, 21$, $j = 1, \ldots, 21$, can be derived using symbolic computation (Maple).

For two stage SVDT tensegrity structures the prestressability conditions are

$$A(q)T = 0 \quad \text{and} \quad T_j > 0 \ \text{for} \ j = 1, \dots, 21, \tag{52}$$

where $A(q)$ is a 21×21 matrix.

As for two stage SVD and SVDB two stage tensegrity structures, we shall consider the symmetrical prestressable configurations class. The generalized coordinates corresponding to a symmetrical configuration can be easily expressed in terms of α, δ, and h, (α, δ, h are defined in the same way as for two stage SVD and SVDB tensegrity structures). These generalized coordinates are given by

$$\begin{aligned}
&\alpha_{11} = \alpha_{22} = \alpha, \qquad \alpha_{21} = \alpha_{32} = \alpha + \frac{4\pi}{3}, \qquad \alpha_{31} = \alpha_{12} = \alpha + \frac{2\pi}{3},\\
&x_{12} = \frac{l}{4}\sin(\delta)\cos(\alpha) + \frac{\sqrt{3}}{4} l \sin(\delta)\sin(\alpha) - \frac{b}{2},\\
&y_{12} = \frac{b}{2\sqrt{3}} - \frac{\sqrt{3}}{4} l \sin(\delta)\cos(\alpha) + \frac{l}{4}\sin(\delta)\sin(\alpha), \qquad z_{12} = \frac{3}{2} l\cos(\delta) - h,\\
&x_{22} = \frac{b}{2} - \frac{l}{2}\sin(\delta)\cos(\alpha), \qquad y_{22} = \frac{b}{2\sqrt{3}} - \frac{l}{2}\sin(\delta)\sin(\alpha), \qquad z_{22} = \frac{3}{2} l\cos(\delta) - h,\\
&x_{32} = \frac{l}{4}\sin(\delta)\cos(\alpha) - \frac{\sqrt{3}}{4} l \sin(\delta)\sin(\alpha),\\
&y_{32} = \frac{l}{4}\sin(\delta)\sin(\alpha) + \frac{\sqrt{3}}{4} l \sin(\delta)\cos(\alpha) - \frac{b}{\sqrt{3}},\\
&z_{32} = \frac{3}{2} l\cos(\delta) - h, \qquad \delta_{ij} = \delta, \quad i = 1,2,3, \ \ j = 1,2.
\end{aligned} \tag{53}$$

At a symmetrical configuration all saddle, vertical, and diagonal tendons have the same lengths, respectively, given by the same formulas (13)–(15) as for the SVD type. The top tendons have the same length, b.

As for the SVD and SVDB types, in a symmetrical prestressable configuration, α and δ are restricted to $\alpha \in \{[0, 2\pi) - \frac{\pi}{6}\}$ and $\delta \in (0, \frac{\pi}{2})$.

The prestressability conditions, $A(\alpha, \delta, h)T = 0$, $j = 1, \dots, 21$, have been numerically investigated as for the SVD and SVDB types as follows. For fixed l, b, α, δ the *necessary* prestressability condition, $\det(A(\alpha, \delta, h)) = 0$, has been solved for h and the kernel of the corresponding matrix $A(\alpha, \delta, h)$ has been computed by solving $A(\alpha, \delta, h)T = 0$ for T. We ascertained that the prestressability conditions have atmost one solution for h such that $T_j > 0$ for $j = 1, \dots, 21$, and that, in this case, the tensions in the saddle, vertical, diagonal, and top tendons are respectively equal.Consequently we impose the condition that the tensions in the saddle, vertical, diagonal, and top tendons are respectively equal to T_S, T_V, T_D, and T_T such that the vector of tensions has the following structure:

$$T = [T_D \ T_V \ T_D \ T_V \ T_D \ T_V \ T_S \ T_S \ T_S \ T_S \ T_S \ T_S \ T_V \ T_D \ T_V \ T_D \ T_V \ T_D \ T_T \ T_T \ T_T]^T. \tag{54}$$

If we substitute Eqs. (53) and (54) into Eq. (52) we obtain the reduced prestressability conditions

$$A_r T_r = 0, \quad T_S > 0, \quad T_V > 0, \quad T_D > 0, \quad T_T > 0, \tag{55}$$

where A_r is a 21×4 matrix, and $T_r = [T_S \ T_V \ T_D \ T_T]^T$.

Further investigations showed that the only rows of A_r which might be independent are 1, 2, 3, and 4. The prestressability conditions corresponding to these rows of A_r yield

$$A_e T_r = 0, \quad T_S > 0, \quad T_V > 0, \quad T_D > 0, \quad T_T > 0, \tag{56}$$

where the structure of A_e is

$$A_{\mathrm{e}} = \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ * & 0 & * & 0 \end{bmatrix} \tag{57}$$

with the nonzero elements of A_{e} given by

$$A_{e_{11}} = \frac{1}{S}\left(l\cos(\delta)\sin(\delta) - 2h\sin(\delta) - \frac{b}{\sqrt{3}}\cos(\delta)\cos\left(\alpha - \frac{\pi}{6}\right)\right), \qquad A_{e_{12}} = -\frac{lb}{V}\cos(\delta)\cos\left(\alpha - \frac{\pi}{3}\right),$$

$$A_{e_{13}} = \frac{l}{D}\left(2l\sin(\delta)\cos(\delta) - h\sin(\delta) - \frac{b}{\sqrt{3}}\sin(\alpha)\cos(\delta)\right), \qquad A_{e_{21}} = -\frac{lb\sin(\delta)\cos(\alpha + \frac{\pi}{3})}{\sqrt{3}S},$$

$$A_{e22} = -\frac{lb\sin(\delta)\cos(\alpha + \frac{\pi}{6})}{V}, \qquad A_{e23} = -\frac{lb\sin(\delta)\cos(\alpha)}{\sqrt{3}D},$$

$$A_{e31} = \frac{2l\sin(\delta)\cos(\alpha - \frac{\pi}{3}) - b}{2S}, \qquad A_{e32} = \frac{1}{2V}(2\sqrt{3}l\sin(\delta)\sin(\alpha) - 3b),$$

$$A_{e33} = -\frac{b + 2l\sin(\delta)\cos(\alpha + \frac{\pi}{3})}{2D}, \qquad A_{e34} = -\frac{3}{2},$$

$$A_{e41} = -\frac{2h}{S}, \qquad A_{e43} = \frac{2(l\cos(\delta) - h)}{D}.$$

We next give another important result of this paper.

Theorem 3. *For $\alpha \in \{[0, 2\pi) - \frac{\pi}{6}\}$ and $\delta \in (0, \frac{\pi}{2})$. the set of symmetrical prestressable configurations of two stage SVDT tensegrity structures for which the tensions in the saddle, vertical, diagonal, and top tendons are respectively equal is identical with the set of symmetrical prestressable configurations of two stage SVD tensegrity structures for which the tensions in the saddle, vertical, and diagonal tendons are respectively equal.*

Proof. See Appendix C. □

The forces in all bars are equal and can be computed using Eqs. (33)–(35).

5.2. Two stage SDT tensegrity structures

The last example of a class of tensegrity structures for which the prestressability conditions can be analytically solved is that of two stage SDT tensegrity structures.

A two stage SDT tensegrity structure can be obtained by removing the vertical tendons from a two stage SVDT one. In the same way as for the SD type, we ascertain that two stage SDT tensegrity structures can yield symmetrical prestressable configurations for which the saddle, diagonal, and top tendon tensions are respectively equal and that the necessary condition for this to happen is given by Eq. (36) (which is equivalent to $T_{\mathrm{V}} = 0$).

The symmetrical prestressable configurations for which the saddle, diagonal, and top tendon tensions are respectively equal of two stage SDT tensegrity structures are limit cases of the symmetrical prestressable configurations for which the saddle, vertical, diagonal, and boundary tendon tensions are respectively equal of two stage SVDT tensegrity structures, all the necessary formulas (describing the equilibrium solutions and the state of stress) being obtained from the ones corresponding to the two stage SVDT type by enforcing the condition that the tensions in the vertical tendons are 0.

6. Conclusions

General prestressability conditions for tensegrity structures have been derived through the application of the principle of virtual work. These conditions are expressed as a set of nonlinear equations and inequalities.

The prestressability conditions have been investigated for certain tensegrity structures. For particular classes of prestressable configurations analytical solutions of the prestressability conditions have been found. The state of stress of the structures in these prestressable configurations has been shown to depend only on one scalar parameter, the pretension coefficient. The set of these prestressable configurations can be represented by a surface or by a curve in a certain three dimensional space.

Appendix A. Proof of Theorem 1

We shall first prove that, for given α and δ, a two stage SVD tensegrity structure might yield at most one symmetrical prestressable configuration for which the tensions in the saddle, vertical, and diagonal tendons are respectively equal and that for this to happen conditions (23) are necessary.

We have seen that necessary and sufficient conditions for the two stage SVD tensegrity structure to yield symmetrical prestressable configurations for which the tensions in the saddle, vertical, and diagonal tendons are respectively equal are given by Eq. (21). Consider the third equation in Eq. (21):

$$T_S \frac{h}{S} + T_D \frac{h - l\cos(\delta)}{D} = 0. \tag{A.1}$$

Because $0 < \delta < \frac{\pi}{2}$, $S > 0$, and $D > 0$, conditions $T_S > 0$ and $T_D > 0$ can be simultaneously satisfied only if

$$0 < h < l\cos(\delta). \tag{A.2}$$

Since A_e is a square matrix, conditions (21) require that $\det(A_e) = 0$. This reduces to

$$h^2 u + h\left(\frac{b}{\sqrt{3}} - lu\right)\cos(\delta) + l\left(lu - \frac{b}{2\sqrt{3}}\right)\cos^2(\delta) = 0, \tag{A.3}$$

where $u = \sin(\delta)\cos(\alpha + \frac{\pi}{6})$.

Consider now Eq. (A.3) and condition $0 < h < l\cos(\delta)$. For $\alpha = \frac{\pi}{3}$ Eq. (A.3) is linear in h and has only one solution, $h = l\cos(\delta)/2$, which satisfies $0 < h < l\cos(\delta)$. Assume that $\alpha \neq \frac{\pi}{3}$. Then Eq. (A.3) is quadratic in h and, since $0 < \delta < \frac{\pi}{2}$, has real solutions for h if and only if

$$\frac{b^2}{3} - 3l^2u^2 \geqslant 0. \tag{A.4}$$

Consider the following solution for h of Eq. (A.3):

$$h = h_+ = \frac{\cos(\delta)}{2u}\left(lu + \sqrt{\frac{b^2}{3} - 3l^2u^2} - \frac{b}{\sqrt{3}}\right). \tag{A.5}$$

Condition $0 < h < l\cos(\delta)$ is equivalent to $|h - \frac{l\cos(\delta)}{2}| < \frac{l\cos(\delta)}{2}$, which, for $h = h_+$ becomes

$$\left|\sqrt{\frac{b^2}{3} - 3l^2u^2} - \frac{b}{\sqrt{3}}\right| < l\sin(\delta)\left|\cos\left(\alpha + \frac{\pi}{6}\right)\right|. \tag{A.6}$$

After raising Eq. (A.6) to the second power, and after several algebraic manipulations, we get the equivalent condition

$$\frac{b}{\sqrt{3}}\sqrt{\frac{b^2}{3}-3l^2u^2} > \frac{b^2}{3}-2l^2u^2. \tag{A.7}$$

Because of Eq. (A.4), the right-hand side of this inequality is always positive. After raising Eq. (A.4) to the second power and performing several algebraic manipulations, we get the equivalent condition

$$l\sin(\delta)\left|\cos\left(\alpha+\frac{\pi}{6}\right)\right| < \frac{b}{2\sqrt{3}} \tag{A.8}$$

which is stronger than Eq. (A.4) (if Eq. (A.8) is satisfied then Eq. (A.4) is also satisfied).

The other solution for h of Eq. (A.3), let it be called h_-, does not satisfy $0 < h_- < l\cos(\delta)$. This can be easily proved in the same manner as it has been proved that $0 < h_+ < l\cos(\delta)$.

Thus for given $\alpha \in [0, 2\pi)$ and $\delta \in (0, \frac{\pi}{2})$ Eq. (A.3) might have at most one solution satisfying Eq. (A.2):

$$h = \begin{cases} \frac{\cos(\delta)}{2u}\left(lu+\sqrt{\frac{b^2}{3}-3l^2u^2}-\frac{b}{\sqrt{3}}\right) & \text{if } \alpha \neq \frac{\pi}{3}, \\ \frac{l\cos(\delta)}{2} & \text{if } \alpha = \frac{\pi}{3}. \end{cases} \tag{A.9}$$

The necessary and sufficient condition for this solution to satisfy $0 < h < l\cos(\delta)$ is

$$l\sin(\delta)\left|\cos\left(\alpha+\frac{\pi}{6}\right)\right| < \frac{b}{2\sqrt{3}} \tag{A.10}$$

with

$$h = 0 \quad \text{for } l\sin(\delta)\cos\left(\alpha+\frac{\pi}{6}\right) = \frac{b}{2\sqrt{3}} \tag{A.11}$$

and

$$h = l\cos(\delta) \quad \text{for } l\sin(\delta)\cos\left(\alpha+\frac{\pi}{6}\right) = -\frac{b}{2\sqrt{3}}. \tag{A.12}$$

Consider a triple (α, δ, h) which satisfies Eq. (A.2) and Eq. (A.3). Then, the kernel of A_e is one dimensional. Indeed, the rank of the matrix composed of the first two columns of A_e is 2 because

$$0 < h < l\cos(\delta) \quad \text{and} \quad 0 < \delta < \frac{\pi}{2}. \tag{A.13}$$

The tensions in the saddle, vertical, and diagonal tendons can then be expressed as

$$T_r = PT_0 \tag{A.14}$$

where P is an arbitrary scalar called the *pretension coefficient* and T_0 is a normalized basis of $\mathrm{Ker}(A_e)$ given by

$$T_0^T = [T_{0_S}\ T_{0_V}\ T_{0_D}] = \frac{[T_S^r\ T_V^r\ T_D^r]}{\sqrt{6}\,\|[T_S^r\ T_V^r\ T_D^r]\|} \tag{A.15}$$

where

$$T_V^r = \begin{cases} \frac{V}{D}\frac{1}{\sqrt{3}\cos\left(\alpha+\frac{\pi}{6}\right)}\left(\left(\frac{l\cos(\delta)}{h}-1\right)\sin\left(\alpha-\frac{\pi}{6}\right)-\cos(\alpha)\right)T_D^r & \text{if } \alpha \neq \frac{\pi}{3}, \\ \frac{VT_D^r}{D}\left(\frac{3l}{2b}\sin(\delta)-1\right) & \text{if } \alpha = \frac{\pi}{3}, \end{cases} \tag{A.16}$$

$$T_S^r = \begin{cases} \frac{S}{D}\left(\frac{l\cos(\delta)}{h} - 1\right)T_D^r & \text{if } \alpha \neq \frac{\pi}{3}, \\ T_D^r & \text{if } \alpha = \frac{\pi}{3}, \end{cases} \tag{A.17}$$

$$T_D^r = 1. \tag{A.18}$$

The normalization has been performed such that the Euclidean norm of the vector of all 18 tensions is one for $P = 1$. These formulas are valid if and only if Eq. (A.10) holds and h is given by Eq. (A.9).

We now investigate the condition that the tensions are positive. Condition $T_D > 0$, and $T_D = PT_{0_D}$ where $T_{0_D} > 0$ yields $P > 0$. Now, taking into account that $T_D > 0$, $S > 0$, $D > 0$, and Eqs. (A.1) and (A.2) we get that $T_S > 0$. Consider condition $T_V > 0$, which, since $P > 0$, is equivalent to $T_V^r > 0$. Assume that $\alpha \neq \frac{\pi}{3}$. Taking into account Eq. (A.9), $V > 0$, $D > 0$, and the formula for T_V^r, after several algebraic manipulations, condition $T_V^r > 0$ can be shown to be equivalent to

$$\sqrt{\frac{b^2}{3} - 3l^2\sin^2(\delta)\cos^2\left(\alpha + \frac{\pi}{6}\right)}\sin\left(\alpha + \frac{\pi}{6}\right) < \frac{b}{\sqrt{3}}\sin\left(\alpha + \frac{\pi}{6}\right) - \sqrt{3}l\sin(\delta)\cos^2\left(\alpha + \frac{\pi}{6}\right). \tag{A.19}$$

Assume first that $\sin(\alpha + \frac{\pi}{6}) < 0$. Then, the right-hand side of Eq. (A.19) is negative. After raising Eq. (A.19) to the second power and after performing several algebraic manipulations, we obtain the equivalent condition $3l\sin(\delta) < 2b\sin(\alpha + \frac{\pi}{6})$, which cannot hold since $0 < \delta < \frac{\pi}{2}$.

Assume now that $\sin(\alpha + \frac{\pi}{6}) \geqslant 0$. Then the right-hand side of Eq. (A.19) has to be positive, a condition which leads to

$$\sin\left(\alpha + \frac{\pi}{6}\right) > \sqrt{1 + \frac{b^2}{36l^2\sin^2(\delta)}} - \frac{b}{6l\sin(\delta)}. \tag{A.20}$$

After raising Eq. (A.19) to the second power and performing several algebraic manipulations we get the following condition:

$$3l\sin(\delta) > 2b\sin\left(\alpha + \tfrac{\pi}{6}\right). \tag{A.21}$$

If $\alpha = \frac{\pi}{3}$, $T_V^r > 0$ leads to $3l\sin(\delta) > 2b$ which is a particular case of Eq. (A.21).

Thus we have proved that $T_V^r > 0$ is equivalent to Eqs. (A.20) and (A.21). In addition we have of course to consider condition (A.10). Since Eq. (A.20) implies that $\sin(\alpha + \frac{\pi}{6}) > 0$, Eq. (A.10) becomes

$$\sin\left(\alpha + \frac{\pi}{6}\right) > \sqrt{1 - \frac{b^2}{12l^2\sin^2(\delta)}}. \tag{A.22}$$

We ascertain that for Eqs. (A.20) and (A.21) to simultaneously hold we must have

$$3l\sin(\delta) > \sqrt{2}b \tag{A.23}$$

which implies

$$\sqrt{1 + \frac{b^2}{36l^2\sin^2(\delta)}} - \frac{b}{6l\sin(\delta)} < \sqrt{1 - \frac{b^2}{12l^2\sin^2(\delta)}}. \tag{A.24}$$

If we now consider Eqs. (A.22) and (A.24) we get

$$\sqrt{1 + \frac{b^2}{36l^2\sin^2(\delta)}} - \frac{b}{6l\sin(\delta)} < \sqrt{1 - \frac{b^2}{12l^2\sin^2(\delta)}} < \sin\left(\alpha + \frac{\pi}{6}\right) \tag{A.25}$$

showing that if Eqs. (A.21)–(A.23) hold then Eq. (A.20) is satisfied.

Thus, so far we have proved that for symmetrical prestressable configurations for which the tensions in the saddle, vertical, and diagonal tendons are respectively equal, to exist, conditions (A.21)–(A.23) are necessary (for $\alpha \in \{[0, 2\pi) - \frac{\pi}{6}\}$ and $\delta \in (0, \frac{\pi}{2})$). These conditions imply that α must satisfy $\alpha \in (\frac{\pi}{6}, \frac{\pi}{2})$ This can be easily proved as follows. Assume first that $\sqrt{2}b < 3l\sin(\delta) < 2b$. In order for the inequalities (A.21) and (A.22) to simultaneously hold we must have

$$\sqrt{1 - \frac{b^2}{12l^2\sin^2(\delta)}} < \frac{3l\sin(\delta)}{2b} \tag{A.26}$$

which yields $3l\sin(\delta) > \sqrt{3}b$. Taking into account this condition and Eq. (A.22) we get

$$\frac{\sqrt{3}}{2} < \sqrt{1 - \frac{b^2}{12l^2\sin^2(\delta)}} < \sin\left(\alpha + \frac{\pi}{6}\right). \tag{A.27}$$

If $3l\sin(\delta) \geqslant 2b$, then Eqs. (A.21) and (A.23) are satisfied while Eq. (A.22) yields $\sin(\alpha + \frac{\pi}{6}) > \frac{\sqrt{13}}{4}$. Thus $\sin(\alpha + \frac{\pi}{6}) > \frac{\sqrt{3}}{2}$, which yields $\alpha \in (\frac{\pi}{6}, \frac{\pi}{2})$.

We shall now prove that conditions (23) are also sufficient. The necessary and sufficient conditions for a two stage SVD tensegrity structure to yield symmetrical prestressable configurations for which the tensions in the saddle, vertical, and diagonal tendons are respectively equal, are given by Eq. (21). Hence we have to prove that if Eq. (23) holds then Eq. (21) have at least one solution. Let (α, δ) be a pair which satisfies Eq. (23) and assume that h is given by Eq. (24). As we have seen in the proof of the first part of the theorem, if α and δ satisfy Eq. (23) and h is given by Eq. (24), then h is real valued (actually $0 < h < l\cos(\delta)$) and $\det(A_e) = 0$. Thus $A_e T_r = 0$ has nonzero solutions for T_r. In this case, since $0 < \delta < \frac{\pi}{2}$ and $0 < h < l\cos(\delta)$, the rank of A_e is 2 and the nonzero solutions for T_r are given by Eqs. (26)–(30), as we have already seen. If we now choose any $P > 0$, then all we have to prove is that Eq. (23) implies that $T_S^r > 0$, $T_V^r > 0$, and $T_D^r > 0$. It is easy to see that $T_D^r = 1 > 0$, and $T_S^r > 0$. We have also seen in the first part of the proof of the theorem that $T_V^r > 0$ is equivalent to Eqs. (A.21)–(A.23). We shall now prove that Eq. (23) implies Eqs. (A.21)–(A.23). Indeed, from Eq. (23) we have $\frac{\pi}{6} < \alpha < \frac{\pi}{2}$, from which we get $3l\sin(\delta) > 2b\sin(\alpha + \frac{\pi}{6}) > \sqrt{3}b > \sqrt{2}b$ which proves that Eq. (A.23) holds. Also, since $\frac{\pi}{6} < \alpha < \frac{\pi}{2}$, the third condition from Eq. (23), $l\sin(\delta)|\cos(\alpha + \frac{\pi}{6})| < \frac{b}{2\sqrt{3}}$, implies Eq. (A.22). Lastly, Eq. (A.21) implies itself.

This proves that conditions (23) are also sufficient for the two stage tensegrity SVD tensegrity structures to yield symmetrical prestressable configurations for which the tensions in the saddle, vertical, and diagonal tendons are respectively equal and concludes the proof of the theorem.

Appendix B. Proof of Theorem 2

The set of symmetrical prestressable configurations of two stage SVDB tensegrity structures for which the tensions in the saddle, vertical, diagonal, and boundary tendons are respectively equal is characterized by Eq. (48). The set of symmetrical prestressable configurations of two stage SVD tensegrity structures for which the tensions in the saddle, vertical, and diagonal tendons are respectively equal is characterized by Eq. (21). We shall prove that condition (48) yield the same solution for the overlap in terms of α and δ as Eq. (21) and that this solution exists if and only if Eq. (23) hold.

The third equation in Eq. (48) yields

$$T_S\frac{h}{S} + T_D\frac{h - l\cos(\delta)}{D} = 0 \tag{B.1}$$

showing that T_S and T_D can simultaneously be positive (as required) if and only if $0 < h < l\cos(\delta)$ (since $0 < \delta < \frac{\pi}{2}$, $S > 0$, and $D > 0$).

Prestressability implies that $A_e T_r = 0$ in Eq. (48) must have nonzero solutions for T_r, which leads to the condition that the determinant of A_e is 0, yielding

$$h^2 u + h\left(\frac{b}{\sqrt{3}} - lu\right)\cos(\delta) + l\left(lu - \frac{b}{2\sqrt{3}}\right)\cos^2(\delta) = 0, \tag{B.2}$$

where $u = \sin(\delta)\cos(\alpha + \frac{\pi}{6})$. This equation is exactly the same equation which characterizes the symmetrical prestressable configurations for which the saddle, vertical, and diagonal tendon tensions are respectively equal of two stage SVD tensegrity structures. The equation has to be solved subject to the same conditions $(0 < h < l\cos(\delta), 0 < \delta < \frac{\pi}{2})$, thus the solution for h is the same:

$$h = \begin{cases} \frac{\cos(\delta)}{2u}\left(lu + \sqrt{\frac{b^2}{3} - 3l^2u^2} - \frac{b}{\sqrt{3}}\right) & \text{if } \alpha \neq \frac{\pi}{3}, \\ \frac{l\cos(\delta)}{2} & \text{if } \alpha = \frac{\pi}{3}. \end{cases} \tag{B.3}$$

This solution satisfies $0 < h < l\cos(\delta)$ if and only if $l\sin(\delta)|\cos(\alpha + \frac{\pi}{6})| < \frac{b}{\sqrt{3}}$.

The rank of A_e at such a configuration is investigated next. The determinant of the matrix formed by the intersection of rows 1, 2, 3, and columns 2, 3, 4 of A_e is equal to $\frac{48l}{VD}(h - l\cos(\delta))\sin(\delta)\cos(\alpha + \frac{\pi}{6})$. It is easily seen that, since $0 < h < l\cos(\delta)$ and $0 < \delta < \frac{\pi}{2}$, this determinant can be zero for $\cos(\alpha + \frac{\pi}{6}) = 0$, yielding $\alpha = \frac{\pi}{3}$ or $\alpha = \frac{4\pi}{3}$. However the determinant of the matrix formed by the intersection of rows 2, 3, 4 and columns 2, 3, 4 of A_e is equal to $\frac{8\sqrt{3}}{VD}(h - l\cos(\delta))(\sqrt{3}l\sin(\alpha - \frac{\pi}{3})\sin(\delta)\cos(\alpha) - b\cos(\delta - \frac{\pi}{3}))\cos(\delta)$, which is clearly nonzero for $\alpha = \frac{\pi}{3}$ and for $\alpha = \frac{4\pi}{3}$ (since $0 < h < l\cos(\delta)$ and $0 < \delta < \frac{\pi}{2}$). These facts show that at such a configuration the rank of A_e is 3, thus the kernel of A_e is one dimensional, leading to

$$T_r = T_0 P. \tag{B.4}$$

Here P is an arbitrary positive scalar called the pretension coefficient and T_0 is a normalized basis of the kernel of A_e, given by

$$T_0^T = [T_{0_S}\ T_{0_V}\ T_{0_D}\ T_{0_B}] = \frac{[T_S^r\ T_V^r\ T_D^r\ T_B^r]}{\sqrt{6}\ \|[T_S^r\ T_V^r\ T_D^r\ T_B^r]\|} \tag{B.5}$$

where

$$T_V^r = \begin{cases} \frac{V}{D}\frac{1}{\sqrt{3}\cos\left(\alpha + \frac{\pi}{6}\right)}\left(\left(\frac{l\cos(\delta)}{h} - 1\right)\sin(\alpha - \frac{\pi}{6}) - \cos(\alpha)\right)T_D^r & \text{if } \alpha \neq \frac{\pi}{3}, \\ \frac{VT_D^r}{D}\left(\frac{3l}{2b}\sin(\delta) - 1\right) & \text{if } \alpha = \frac{\pi}{3}, \end{cases} \tag{B.6}$$

$$T_S^r = \begin{cases} \frac{S}{D}\left(\frac{l\cos(\delta)}{h} - 1\right)T_D^r & \text{if } \alpha \neq \frac{\pi}{3}, \\ T_D^r & \text{if } \alpha = \frac{\pi}{3}, \end{cases} \tag{B.7}$$

$$T_B^r = \begin{cases} \frac{T_D^r}{6D}\frac{3l^2\sin(\delta)\cos(\delta) + 6bh\cos(\alpha - \frac{\pi}{3}) - 6lh\sin(\delta) - 2\sqrt{3}bl\cos(\delta)\sin(\alpha)}{\sqrt{3}h\cos(\alpha + \frac{\pi}{6})} & \text{if } \alpha \neq \frac{\pi}{3} \\ \frac{T_D^r}{6D}\frac{2b^2 - 9lb\sin(\delta) + 9l^2\sin^2(\delta)}{b} & \text{if } \alpha = \frac{\pi}{3}, \end{cases} \tag{B.8}$$

$$T_D^r = 1. \tag{B.9}$$

The normalization has been performed such that the Euclidean norm of the vector of all 24 tensions is one for $P = 1$. We ascertain that T_V^r and T_S^r are given by the same formulas like their counterparts for two stage SVD tensegrity structures yielding symmetrical prestressable configurations for which the tensions in the saddle, vertical, and diagonal tendons are respectively equal.

We now investigate the conditions that the tensions are positive. We remark that the expressions of T_S^r, T_V^r, and T_D^r are the same as the ones we obtained for the symmetrical prestressable configurations of two stage SVD tensegrity structures for which the tensions in the saddle, vertical, and diagonal tendons are respectively equal. We have already seen that the solution for the overlap is also the same. Thus, if we consider conditions $T_S > 0$, $T_V > 0$, and $T_D > 0$ for the SVDB type we obtain the same conclusions as for the SVD type, namely that these conditions hold if and only if $P > 0$ and Eq. (23) hold. The proof is identical with the proof of Theorem 1. However, in addition to these conditions, for the SVDB type we have to investigate $T_B > 0$. We shall prove that if Eq. (23) hold then $T_B > 0$.

Consider first that $\alpha \neq \frac{\pi}{3}$. Since $h > 0$, $D > 0$, $T_D^r > 0$, condition $T_B > 0$ is equivalent to

$$\frac{3l^2\sin(\delta)\cos(\delta) + 6bh\cos(\alpha - \frac{\pi}{3}) - 6lh\sin(\delta) - 2\sqrt{3}bl\cos(\delta)\sin(\alpha)}{\cos(\alpha + \frac{\pi}{6})} > 0. \tag{B.10}$$

Substituting h with its value for $\alpha \neq \frac{\pi}{3}$,

$$h = \frac{\cos(\delta)}{2u}\left(lu + \sqrt{\frac{b^2}{3} - 3l^2u^2} - \frac{b}{\sqrt{3}}\right), \quad \text{where } u = \sin(\delta)\cos\left(\alpha + \tfrac{\pi}{6}\right), \tag{B.11}$$

and taking into account that $0 < \delta < \frac{\pi}{2}$, after some algebraic manipulations, condition (B.10) becomes

$$\sin(\delta)\cos^2\left(\alpha + \frac{\pi}{6}\right) + \left(\sin(\delta) - \frac{b}{l}\sin\left(\alpha + \frac{\pi}{6}\right)\right)\left(1 - \sqrt{1 - 9\frac{l^2}{b^2}\sin^2(\delta)\cos^2\left(\alpha + \frac{\pi}{6}\right)}\right) > 0. \tag{B.12}$$

Multiplication of Eq. (B.12) with $1 + \sqrt{1 - 9\frac{l^2}{b^2}\sin^2(\delta)\cos^2(\alpha + \frac{\pi}{6})}$ and further algebraic manipulations lead to

$$-\sqrt{1 - 9\frac{l^2}{b^2}\sin^2(\delta)\cos^2\left(\alpha + \frac{\pi}{6}\right)} < 1 + 9\frac{l^2}{b^2}\sin(\delta)\left(\sin(\delta) - \frac{b}{l}\sin\left(\alpha + \frac{\pi}{6}\right)\right). \tag{B.13}$$

The right-hand side of Eq. (B.13) is a quadratic form in $x = \frac{3l\sin(\delta)}{b}$. Let it be called $R(x)$:

$$R(x) = x^2 - 3x\sin\left(\alpha + \tfrac{\pi}{6}\right) + 1. \tag{B.14}$$

The sign of $R(x)$ is analyzed next. Consider the discriminant, Δ, of the quadratic equation in x, $R(x) = 0$: $\Delta = 9\sin^2(\alpha + \frac{\pi}{6}) - 4$. Because $\frac{\pi}{6} < \alpha < \frac{\pi}{2}$, we have $\Delta > 0$ and $R(x) = 0$ has two different roots. The sign of $R(x)$ is characterized as follows:

- $R(x) \leqslant 0$ if
$$\frac{3\sin(\alpha + \frac{\pi}{6}) - \sqrt{\Delta}}{2} \leqslant x \leqslant \frac{3\sin(\alpha + \frac{\pi}{6}) + \sqrt{\Delta}}{2}. \tag{B.15}$$
- $R(x) > 0$ otherwise.

We now take into account the fact that, from Eq. (23),

$$x = \frac{3l\sin(\delta)}{b} > 2\sin\left(\alpha + \tfrac{\pi}{6}\right) \quad \text{and} \quad \sin\left(\alpha + \tfrac{\pi}{6}\right) > \frac{\sqrt{3}}{2}.$$

It can easily be seen that

$$\frac{3\sin(\alpha + \frac{\pi}{6}) - \sqrt{\Delta}}{2} < 2\sin\left(\alpha + \frac{\pi}{6}\right) < \frac{3\sin(\alpha + \frac{\pi}{6}) + \sqrt{\Delta}}{2}. \tag{B.16}$$

Thus $R(x) > 0$ for

$$x > \frac{3\sin(\alpha + \frac{\pi}{6}) + \sqrt{\Delta}}{2} \tag{B.17}$$

and $R(x) \leqslant 0$ for

$$2\sin\left(\alpha + \frac{\pi}{6}\right) < x \leqslant \frac{3\sin(\alpha + \frac{\pi}{6}) + \sqrt{\Delta}}{2}. \tag{B.18}$$

Now, if $R(x) > 0$ the inequality (B.13) – and thus $T_B > 0$ – is satisfied.

In the second case ($R(x) \leqslant 0$), after raising to the second power the inequality (B.13) and performing some algebraic manipulations, we get the equivalent condition

$$0 > 9\frac{l\sin(\delta)}{b}\left(\frac{l\sin(\delta)}{b} - \sin\left(\alpha + \frac{\pi}{6}\right)\right)^2 - \frac{l\sin(\delta)}{b}\sin^2\left(\alpha + \frac{\pi}{6}\right) + 3\frac{l\sin(\delta)}{b} - 2\sin\left(\alpha + \frac{\pi}{6}\right) \tag{B.19}$$

which, after introducing $x = 3l\sin(\delta)/b$, can be written as

$$0 > \left(x - 2\sin\left(\alpha + \frac{\pi}{6}\right)\right)\left(x^2 - 4x\sin\left(\alpha + \frac{\pi}{6}\right) + 3\right). \tag{B.20}$$

Since $\sin(\alpha + \frac{\pi}{6}) > \frac{\sqrt{3}}{2}$ we have $4\sin^2(\alpha + \frac{\pi}{6}) - 3 > 0$. Then the above inequality becomes

$$0 > \left(x - 2\sin\left(\alpha + \tfrac{\pi}{6}\right)\right)(x - x_1)(x - x_2) \tag{B.21}$$

with:

$$x_1 = 2\sin\left(\alpha + \frac{\pi}{6}\right) - \sqrt{4\sin^2\left(\alpha + \frac{\pi}{6}\right) - 3}, \qquad x_2 = 2\sin\left(\alpha + \frac{\pi}{6}\right) + \sqrt{4\sin^2\left(\alpha + \frac{\pi}{6}\right) - 3}. \tag{B.22}$$

Since $x > x_1$ (as it can be easily seen) and $x - 2\sin(\alpha + \frac{\pi}{6}) > 0$ the above condition reduces to:

$$x < x_2. \tag{B.23}$$

Now, recall that

$$R(x) \leqslant 0 \quad \text{if } 2\sin\left(\alpha + \tfrac{\pi}{6}\right) < x \leqslant \frac{3\sin\left(\alpha + \frac{\pi}{6}\right) + \sqrt{\Delta}}{2}.$$

Analyzing the position of x_2 with respect to $(3\sin(\alpha + \frac{\pi}{6}) + \sqrt{\Delta})/2$ we reach the following conclusions.

- If $\sin\left(\alpha + \frac{\pi}{6}\right) > \frac{2}{\sqrt{5}}$ then $\frac{3\sin\left(\alpha + \frac{\pi}{6}\right) + \sqrt{\Delta}}{2} < x_2$ thus $x - x_2 < 0$ and $T_B > 0$.
- If $\frac{\sqrt{3}}{2} < \sin\left(\alpha + \frac{\pi}{6}\right) \leqslant \frac{2}{\sqrt{5}}$ and $2\sin\left(\alpha + \frac{\pi}{6}\right) < x < x_2$ then $T_B > 0$.
- If $\frac{\sqrt{3}}{2} < \sin\left(\alpha + \frac{\pi}{6}\right) \leqslant \frac{2}{\sqrt{5}}$ and $x_2 \leqslant x \leqslant \frac{3\sin\left(\alpha + \frac{\pi}{6}\right) + \sqrt{\Delta}}{2}$ then $T_B \leqslant 0$.

Next we take into account the third inequality in Eq. (23),

$$l\sin(\delta)\left|\cos\left(\alpha + \frac{\pi}{6}\right)\right| < \frac{b}{2\sqrt{3}}. \tag{B.24}$$

We shall prove that the conditions leading to $T_B \leqslant 0$,

$$\frac{\sqrt{3}}{2} < \sin\left(\alpha+\frac{\pi}{6}\right) \leqslant \frac{2}{\sqrt{5}} \text{ and } x_2 \leqslant x \leqslant \frac{3\sin\left(\alpha+\frac{\pi}{6}\right)+\sqrt{\Delta}}{2}, \tag{B.25}$$

and $l\sin(\delta)|\cos(\alpha+\frac{\pi}{6})| < \frac{b}{2\sqrt{3}}$ cannot simultaneously hold. Indeed, $\frac{\sqrt{3}}{2} < \sin(\alpha+\frac{\pi}{6}) \leqslant \frac{2}{\sqrt{5}}$ and $\frac{\pi}{6} < \alpha < \frac{\pi}{6}$ yield $\frac{\pi}{6} < \alpha \leqslant \arcsin(\frac{2}{\sqrt{5}}) - \frac{\pi}{6}$. In this case $l\sin(\delta)|\cos(\alpha+\frac{\pi}{6})| < \frac{b}{2\sqrt{3}}$ is equivalent to

$$x < \frac{\sqrt{3}}{2\sqrt{1-\sin^2(\alpha+\frac{\pi}{6})}}. \tag{B.26}$$

But, on the other hand

$$x_2 = 2\sin\left(\alpha+\frac{\pi}{6}\right) + \sqrt{4\sin^2\left(\alpha+\frac{\pi}{6}\right)-3} \leqslant x. \tag{B.27}$$

We can prove that, for $\frac{\sqrt{3}}{2} < \sin(\alpha+\frac{\pi}{6}) \leqslant \frac{2}{\sqrt{5}}$, we have

$$\frac{\sqrt{3}}{2\sqrt{1-\sin^2(\alpha+\frac{\pi}{6})}} < x_2. \tag{B.28}$$

Indeed, this inequality is equivalent to

$$\sqrt{3} < 2\sqrt{1-\sin^2\left(\alpha+\frac{\pi}{6}\right)}\left(2\sin\left(\alpha+\frac{\pi}{6}\right) + \sqrt{4\sin^2\left(\alpha+\frac{\pi}{6}\right)-3}\right). \tag{B.29}$$

The right-hand side of this inequality is a function of $s = \sin(\alpha+\frac{\pi}{6})$, $f(s) = 2\sqrt{1-s^2}(2s+\sqrt{4s^2-3})$, which is real valued only for $\frac{\sqrt{3}}{2} < s \leqslant \frac{2}{\sqrt{5}}$. Its range is $(\sqrt{3}, 2]$ as it can be easily checked by computing the derivative of $f(s)$ and ascertaining that it is strictly positive for $\frac{\sqrt{3}}{2} < s < \frac{2}{\sqrt{5}}$ and zero for $s = \frac{2}{\sqrt{5}}$.

But Eqs. (B.26) and (B.28) yield $x < x_2$ which contradicts Eq. (B.27). Thus $l\sin(\delta)|\cos(\alpha+\frac{\pi}{6})| < \frac{b}{2\sqrt{3}}$ and $T_B \leqslant 0$ are not compatible.

This proves that $T_B > 0$ if conditions (23) hold.

For $\alpha = \frac{\pi}{3}$ we can go through the same steps of the above proof using the values of h and T_B for $\alpha = \frac{\pi}{3}$ and reach the conclusion that $T_B > 0$. The proof is simpler and is not given here.

Appendix C. Proof of Theorem 3

The proof is similar with the one of Theorem 2. We shall prove that conditions (56), which characterize the set of symmetrical prestressable configurations of two stage SVDT tensegrity structures for which the tensions in the saddle, vertical, diagonal, and top tendons are respectively equal, yield the same solution for h in terms of α and δ as Eq. (21) and that this solution exists if and only if Eq. (23) hold.

The fourth equation in Eq. (56) yields

$$T_S\frac{h}{S} + T_D\frac{h-l\cos(\delta)}{D} = 0 \tag{C.1}$$

showing that T_S and T_D can simultaneously be positive if and only if $0 < h < l\cos(\delta)$ (since $S > 0$, $D > 0$, and $0 < \delta < \frac{\pi}{2}$).

On the other hand since $A_eT_r = 0$ must have nonzero solutions for T_r and A_e is a square matrix we get the condition that the determinant of A_e is 0, yielding

$$h^2u + h\left(\frac{b}{\sqrt{3}} - lu\right)\cos(\delta) + l\left(lu - \frac{b}{2\sqrt{3}}\right)\cos^2(\delta) = 0, \tag{C.2}$$

where $u = \sin(\delta)\cos(\alpha + \frac{\pi}{6})$. This equation is the same as the one which characterizes the symmetrical prestressable configurations for which the saddle, vertical, and diagonal tendon tensions are respectively equal of two stage SVD tensegrity structures and it has to be solved subject to the same conditions $(0 < h < l\cos(\delta), 0 < \delta < \frac{\pi}{2})$. Hence the solution for h is the same:

$$h = \begin{cases} \frac{\cos\delta}{2u}\left(lu + \sqrt{\frac{b^2}{3} - 3l^2u^2} - \frac{b}{\sqrt{3}}\right) & \text{if } \alpha \neq \frac{\pi}{3} \\ \frac{l\cos(\delta)}{2} & \text{if } \alpha = \frac{\pi}{3}. \end{cases} \tag{C.3}$$

This solution satisfies $0 < h < l\cos(\delta)$ if and only if $l\sin(\delta)|\cos(\alpha + \frac{\pi}{6})| < \frac{b}{2\sqrt{3}}$.

The rank of A_e at such a configuration is investigated next. The determinant of the matrix formed by the intersection of rows 1, 3, 4, and columns 2, 3, 4 of A_e is $[3lb(h - l\cos(\delta))\cos(\delta)\cos(\frac{\pi}{3} - \alpha)]/VD$, and the determinant of the matrix formed by the intersection of rows 2, 3, 4, and columns 2, 3, 4 of A_e is equal to $[3lb(h - l\cos(\delta)\sin(\delta)\sin(\frac{\pi}{3} - \alpha)]/VD$. Since $0 < h < l\cos(\delta)$ and $0 < \delta < \frac{\pi}{2}$ these two determinants cannot be simultaneously 0, thus the rank of A_e is 3. Its kernel is given by

$$T_r = T_0 P. \tag{C.4}$$

Here P is an arbitrary positive scalar called the pretension coefficient and T_0 is a normalized basis of the kernel of A_e, given by

$$T_0^T = [T_{0_S}\ T_{0_V}\ T_{0_D}\ T_{0_T}] = \frac{[T_S^r\ T_V^r\ T_D^r\ T_T^r]}{\sqrt{6}\ \|[T_S^r\ T_V^r\ T_D^r\ T_T^r]\|}, \tag{C.5}$$

where

$$T_V^r = \begin{cases} \frac{V}{D}\frac{1}{\sqrt{3}\cos\left(\alpha+\frac{\pi}{6}\right)}\left(\left(\frac{l\cos(\delta)}{h} - 1\right)\sin\left(\alpha - \frac{\pi}{6}\right) - \cos(\alpha)\right)T_D^r & \text{if } \alpha \neq \frac{\pi}{3}, \\ \frac{VT_D^r}{D}\left(\frac{3l}{2b}\sin(\delta) - 1\right) & \text{if } \alpha = \frac{\pi}{3}, \end{cases} \tag{C.6}$$

$$T_S^r = \begin{cases} \frac{S}{D}\left(\frac{l\cos(\delta)}{h} - 1\right)T_D^r & \text{if } \alpha \neq \frac{\pi}{3}, \\ T_D^r & \text{if } \alpha = \frac{\pi}{3}, \end{cases} \tag{C.7}$$

$$T_T^r = \begin{cases} \frac{T_D^r}{6D}\frac{3l^2\sin(\delta)\cos(\delta)+6bh\cos(\alpha-\frac{\pi}{3})-6lh\sin(\delta)-2\sqrt{3}bl\cos(\delta)\sin(\alpha)}{\sqrt{3}h\cos(\alpha+\frac{\pi}{6})} & \text{if } \alpha \neq \frac{\pi}{3}, \\ \frac{T_D^r}{6D}\frac{2b^2-9lb\sin(\delta)+9l^2\sin^2(\delta)}{b} & \text{if } \alpha = \frac{\pi}{3}, \end{cases} \tag{C.8}$$

$$T_D^r = 1. \tag{C.9}$$

The normalization has been performed such that the Euclidean norm of the vector of all 21 tensions is one for $P = 1$.

We remark that h, T_S^r, T_V^r, T_D^r are given by the same formulas as for the SVDB type, and that T_T^r is given by the same formula as T_B^0 (for the same SVDB type). Hence conditions $T_S^r > 0$, $T_V^r > 0$, $T_D^r > 0$, and $T_T^r > 0$ lead to the same conditions as for the SVDB type, which as shown in Theorem 2 are, at their turn, the same as for the SVD type (given by Eq. (23) with the overlap given by Eq. (24)).

This proves that the set of two stage SVDT tensegrity structures which yield symmetrical prestressable configurations for which the tensions in the saddle, vertical, diagonal, and top tendons are respectively equal is identical with the set of two stage SVDB tensegrity structures which yield symmetrical prestressable configurations for which the tensions in the saddle, vertical, diagonal, and boundary tendons are respectively equal. By Theorem 2 we have that this set is also identical with the set of symmetrical prestressable configurations for which the tensions in the saddle, vertical, and diagonal tendons are respectively equal yielded by two stage SVD tensegrity structures.

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